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Two-matrix model with semiclassical potentials and extended Whitham hierarchy

M Bertola

Department of Mathematics and Statistics, Concordia University 7141 Sherbrooke W, Montréal, QC H4B 1R6, Canada
and
Centre de Recherches Mathématiques, Université de Montréal CP 6128, succ centre ville, Montréal, QC H3C 3J7, Canada

E-mail: bertola@mathstat.concordia.ca

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Abstract
We consider the two-matrix model with potentials whose derivatives are arbitrary rational functions of fixed pole structure and the support of the spectra of the matrices are union of intervals (hard edges). We derive an explicit formula for the planar limit of the free energy and we derive a calculus which allows us to compute derivatives of arbitrarily high order by extending classical Rauch's variational formulae. The four-point correlation functions are explicitly worked out. The formalism extends naturally to the computation of *residue formulae* for the tau function of the so-called universal Whitham hierarchy studied mainly by I Krichever: our setting extends the moduli space in that there are certain extra data.

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1. Introduction

The two-matrix model has recently been investigated from both the point of view of its asymptotic behaviour for large sizes of the matrices [1, 2, 17–19], and in view of the very rich connections to integrable systems (2-Toda lattice) and biorthogonal polynomials [3–5, 27].

We briefly recall that the model consists of pairs of Hermitian matrices of size N with an (unnormalized) probability density of the form

$$\mathrm{d}\mu(M_1, M_2) = \mathrm{d}M_1\, \mathrm{d}M_2 \exp\left[-\frac{1}{\hbar}\mathrm{Tr}\,(V_1(M_1) + V_2(M_2) - M_1M_2)\right], \tag{1.1}$$

$$\mathcal{Z}_N(V_1, V_2, t) := \int \mathrm{d}\mu, \qquad t := N\hbar. \tag{1.2}$$

In most literature the functions V_1, V_2 (called *potentials*) are required to be polynomials (if convergence is an issue) or the formal Taylor series if only formal aspects are considered.

The function (or functional) $\mathcal{Z}_N(V_1, V_2)$, called *partition function*, is one of the foci of interest in modern applications. For fixed N, it is related to the Miwa–Jimbo–Ueno isomonodromic tau function of a certain ODE ([5] for polynomial potentials, or [4] for the similar connection in the case of the one-matrix model). Its logarithm $\mathcal{F}_N := \frac{1}{N^2}\ln\mathcal{Z}_N$, referred to as *free energy*, has particular importance in the $N \to \infty$ regime: indeed a formal manipulation of the integral shows that it admits an expansion in inverse square powers of N and that each term in the expansion is a generating function for the numbers of polyvalent ribbon graphs on topological surfaces (we refer to [10, 14] which contain a comprehensive bibliography).

The present paper deals with the leading term in this expansion

$$\mathcal{F}(V_1, V_2, t) := \lim_{N\to\infty}\frac{1}{N^2}\ln\mathcal{Z}_N, \tag{1.3}$$

where $t = N\hbar$ is kept fixed in the limit process.

This limit, whose existence is a working assumption, is called the *planar limit* in the physical literature because it is related to the enumeration of 'planar' polyvalent ribbon graphs (i.e., graphs that can be drawn on a genus zero surface, namely the (compactified) complex plane).

Such a model in the planar limit was considered in [8] in the 1990s to explain the connection of multicritical regimes and rational (p, q) matter fields [12].

The setting of the present paper is in the spirit of our previous papers [1, 2], where the planar limit of the two-matrix model was considered for polynomial potentials and in an algebro-geometric setting which is, in principle, independent of the physical assumptions. Here we generalize completely that setting to the case of potentials whose derivative is an arbitrary rational function: formally the model is well defined for arbitrary potentials with complex coefficients, provided that we constrain the spectrum to belong to certain contours in the complex plane along the lines explained in [5, 4]. In this case, however, the matrices M_i are no longer Hermitian, but only normal (i.e., commuting with their Hermitian adjoint).

If we insist on a *bona fide* Hermitian model we should impose that V_i are real functions, bounded from below on the real axis.

In addition to these data, we impose that the spectrum contains segments with extrema $\{X_i\}$ for the first matrix and $\{Y_j\}$ for the second matrix (*hard edges* of the spectra): in the case of Hermitian matrices we would be restricting the support of the spectra to some arbitrary union of intervals, for example

$$\mathcal{Z}_N(V_1, V_2, \mathfrak{J}, \mathfrak{K}) := \int_{\mathcal{H}(\mathfrak{J})}\mathrm{d}M_1\int_{\mathcal{H}(\mathfrak{K})}\mathrm{d}M_2\exp\left(-\frac{1}{\hbar}\mathrm{Tr}(V_1(M_1) + V_2(M_2) - M_1M_2)\right). \tag{1.4}$$

Here $\mathcal{H}(\mathfrak{J})$ and $\mathcal{H}(\mathfrak{K})$ stand for the sets of Hermitian matrices whose eigenvalues are in $\mathfrak{J}$ ($\mathfrak{K}$ respectively), assumed to be a finite union of intervals. The potentials V_i have rational derivative with poles outside of the supports of the spectra. The partition function becomes thus a function not only of the potentials, but also of the *hard edges*, namely the endpoints of the multi-intervals $\mathfrak{J}, \mathfrak{K}$.

Some aspects of this model have been analysed in two papers [18] and [4] from two opposite points of view: in [18] were derived the formal properties of the spectral curve and the loop equations in the large N limit, whereas in [4] were considered the properties of the associated biorthogonal polynomials and the differentials equations they satisfy for finite N, together with the certain Riemann–Hilbert data.

For polynomial potentials, the approach using the loop equations (reparametrization invariance) have yielded spectacular results [9, 15, 16] in the study of the formal aspects of the $\frac{1}{N^2}$ expansion. The loop equations show that in the planar limit the resolvents of one of the two matrices

$$W(x) = \lim_{N\to\infty} \frac{1}{N} \left\langle \frac{1}{x - M_1} \right\rangle, \qquad \widetilde{W}(y) = \lim_{N\to\infty} \frac{1}{N} \left\langle \frac{1}{y - M_2} \right\rangle \tag{1.5}$$

satisfy an *algebraic* equation if we replace $y = Y(x) := W(x) - V_1'(x)$. This means that there is a rational expression that defines a (singular) curve in $\Sigma \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$—hereby referred to as *spectral curve*

$$E(x, y) = 0 \tag{1.6}$$

and that the cuts of the branched covers $x : \Sigma \to \mathbb{P}^1$ and $y : \Sigma \to \mathbb{P}^1$ describe the support of the asymptotic density of eigenvalues, and the jumps across these cuts describe the densities themselves.

From the finite N analysis, the spectral curve [5] arises naturally in conjunction with the ODE satisfied by the associated biorthogonal polynomials; indeed any s_2 consecutive biorthogonal polynomials (where s_2 is the total degree of the rational function $V_2'(y)$) satisfy an $(s_2 + 1)$ system of first-order ODEs, namely an equation of the form

$$\partial_x \Psi_N(x) = D_N(x)\Psi_N \tag{1.7}$$

and the spectral curve is nothing, but $E_N(x, y) = \det(y\mathbf{1} - D_N(x)) = 0$. While clearly certain properties are valid only for finite N or in the infinite limit, certain other properties can be read off both regimes: for instance, it can be seen [4] that at the hard edges the matrix $D_N(x)$ has simple poles with nilpotent rank-one residue. This implies certain local structure of the spectral curve $y(x)$ above these points.

In an algebro-geometric approach, the functions x, y themselves are meromorphic functions on the spectral curve Σ with specified pole structure and specified singular part near the poles. The loop equations also provide a first-order overdetermined set of compatible equations for the free energy; these however are not sufficient to uniquely determine the partition function because the polar data of the functions x, y need to be supplemented by extra parameters. This is a purely algebro-geometric consideration, but they also can be heuristically justified along the lines of [7]. It turns out that the extra unspecified parameters can be taken as the contour integrals

$$\epsilon_\gamma := \oint_\gamma y\,\mathrm{d}x, \tag{1.8}$$

over a maximal set of 'independent' non-intersecting contours. The reader with some background in algebraic geometry will recognize that there are $g = \mathrm{genus}(\Sigma)$ such contours[1]. These parameters are often called 'filling fractions' and in principle they be should uniquely determined by the potentials; the loop equations cannot determine the filling fraction, but can determine the variations of the free energy with respect to them. This way one obtains an extended set of (still compatible) PDEs for $\mathcal{F}$ in terms of the full moduli of the algebro-geometric problem: we call this function the *non-equilibrium free energy*. In this situation one can actually integrate the PDEs and provide a formula for the planar limit, $\mathcal{F}$.

Note that the actual free energy of the model is obtained by expressing the filling fraction implicitly as functions of the potentials via the equations

$$\partial_{\epsilon_j}\mathcal{F}(V_1, V_2, t, \underline{\epsilon}) \equiv 0. \tag{1.9}$$

[1] More appropriately one should consider only the imaginary parts of these integrals over the full homology of the curve.

Implicit solution yields $\underline{\epsilon} = \underline{\epsilon}(V_1, V_2, t)$; the resulting function

$$\mathcal{G}(V_1, V_2, t) := \mathcal{F}(V_1, V_2, t, \underline{\epsilon}(V_1, V_2, t)), \tag{1.10}$$

will be called the *equilibrium free energy*. The distinction is important when computing the higher-order derivatives of $\mathcal{G}$, inasmuch as they differ by the higher-order derivatives of $\mathcal{F}$ by virtue of the chain-rule; indeed while

$$\frac{\delta \mathcal{G}}{\delta V_1(x)} = \left.\frac{\delta \mathcal{F}}{\delta V_1(x)}\right|_{\underline{\epsilon}=\underline{\epsilon}(V_1,V_2,t)}, \tag{1.11}$$

(since $\partial_{\epsilon_j}\mathcal{F} = 0$) for the second and higher variations the equations do differ, for example

$$\frac{\delta^2 \mathcal{G}}{\delta V_1(x)\delta V_1(x')} = \left.\left[\frac{\delta^2 \mathcal{F}}{\delta V_1(x)\delta V_1(x')} + \sum_{j=1}^{g} \frac{\delta \partial_{\epsilon_j}\mathcal{F}}{\delta V_1(x)} \frac{\delta \epsilon_j}{\delta V_1(x)}\right]\right|_{\underline{\epsilon}=\underline{\epsilon}(V_1,V_2,t)}. \tag{1.12}$$

We will provide simple formulae for both $\mathcal{G}$, $\mathcal{F}$.

1.1. Connection with the normal matrix model

Although the matrices M_1, M_2 we consider are normal (with spectrum on contours in the complex plane), the denomination of the 'normal matrix model' is traditionally referred to a slightly different, but not a unrelated model. Indeed one then considers the set of normal complex matrices M and the partition function

$$\mathcal{Z}_N^{(NM)} := \int_{\mathcal{N}} \mathrm{d}M \wedge \mathrm{d}M^\dagger \exp\left(-\frac{1}{\hbar}(\mathrm{Tr}\,\Re(V(M)) + \mathrm{Tr}\, MM^\dagger)\right), \tag{1.13}$$

where $\mathcal{N}$ denotes here the space of all complex normal matrices. On a formal level, the Hermitian two-matrix model and the normal matrix model are almost indistinguishable in the large N limit, the latter being a *real section* of the former.

This is particularly evident in the eigenvalue representation of the partition function

$$\mathcal{Z}_N^{(NM)} \propto \int_{\mathbb{C}^N} |\Delta(\mathbf{z})|^2 \exp\left(-\frac{1}{\hbar}\left(\sum_{i=1}^{N} \Re(V(z_i)) + |z_i|^2\right)\right) \prod \mathrm{d}^2 z_i \tag{1.14}$$

compared to the eigenvalue representation of the two-matrix model's case

$$\mathcal{Z}_N \propto \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Delta(\mathbf{x})\Delta(\mathbf{y}) \exp\left(-\frac{1}{\hbar}\left(\sum_{i=1}^{N} V_1(x_i) + V_2(y_i) + x_i\, y_i\right)\right) \prod \mathrm{d}x_i\, \mathrm{d}y_i, \tag{1.15}$$

where Δ denotes the Vandermonde determinant of the given eigenvalues. If we set $V_2 = \overline{V}_1$ and $y_i = \overline{x}_i$ and proceed with formal manipulations, the two integrals become the same[2]. (This situation has been extensively analysed in [21], where the general structure of the spectral curve is studied in both the two-matrix model and the normal matrix model.)

In the large N limit, the partition function of the normal matrix model tends to the 'tau function' of a domain of the complex plane [21, 24] and the deformations of the potential V are identified with the external harmonic moments of the domain [31]. The infinitesimal deformations with respect to these harmonic moments entail connections with the universal Whitham hierarchy [26] as explained in [30].

In this limit the connection between the planar free energy and the above-mentioned tau function of domains is even more stringent. As explained in [1, 2] the spectral curve of the

[2] Of course, the *convergence* of the integral is valid in different regions of the space of potentials, which makes the manipulation at best an expression of some analytic continuation.

two-matrix model reduces to a Shottky-double under certain reality conditions; specifically if the spectral curve has an antiholomorphic involution fixing a maximal number $(g + 1)$ of closed curves (a Harnack-M curve [20]). This real section of the moduli space of the two-matrix-model curves includes the case of simply and multiply connected domains for the normal matrix model.

1.2. Outline of the paper

The main approach of this paper is similar to [1, 2, 19]: namely in section 2 we ascertain the relevant algebro-geometric data in a convenient abstract formulation of the moduli space of spectral data for matrix models. We also recall the definition and properties of the Bergman kernel (section 2.1), which plays an essential role in the paper.

In section 3, we integrate the differential equations determining the planar limit of the free energy: this part gives a much more direct computation compared to [1] and also accounts for the new moduli of the problem (the other poles of the potentials and the hard edges).

We also explain, using mainly the ideas introduced in [2], how to evaluate the 'observables' (partial derivatives) of the free energy up to order 3 in terms of residues involving the Bergman kernel.

In section 4, we develop the formalism for the 'calculus' that allows us to compute arbitrarily high-order partial derivatives; we recall that the derivatives of $\mathcal{F}$ represent higher-order correlators of the spectral invariants of the model in this planar limit and also, the coefficients of their expansion in the parameters of the potentials can be related to enumerative problems of polyvalent fat graph on the sphere. This calculus relies on an extension of Rauch's variational formulae to higher-order variations (usual Rauch's formulae are used to describe first-order variations of the matrix of periods).

In the appendices we report details on the definition of the regularized integrals (appendices A and D) used in the expression of the free energy.

In appendix B we exemplify our moduli space to the case of genus zero curves (usually referred to as the 'one-cut' case, since the support of the spectral densities is a single interval). Moreover, we dwell slightly more on the relationships to the tau function of conformal maps (in this case for simply-connected domains).

Appendix C develops the theory beyond the case of importance to matrix models: in this generalization the spectral functions are replaced by meromorphic differentials much in the spirit of [26] and Seiberg–Witten models. The calculus for higher-order variations is then applied to this new *extended Whitham hierarchy*, providing new residue formulae.

2. Setting and notation

The moduli space of our data is an extension of that in [1][3]. It consists of a (smooth) curve Σ_g of the genus g with $2 + K + L$ distinct-marked points $\infty_{\mathbf{X}}, p_1, \ldots, p_K, \infty_{\mathbf{Y}}, q_1, \ldots, q_L$ and two functions $\mathbf{X}$ and $\mathbf{Y}$ with the following pole structure.

(1) The function $\mathbf{X}$ has the following divisor of poles:

$$(\mathbf{X})_- = \infty_{\mathbf{X}} + d_{2,\infty}\infty_{\mathbf{Y}} + \sum_{\alpha=1}^{H_1}(d_{2,\alpha} + 1)p_\alpha + \sum_{\ell=1}^{K_1}\eta_\ell. \tag{2.1}$$

[3] The functions that there were denoted by P, Q are here denoted by $\mathbf{Y}$, $\mathbf{X}$.

(2) The function $\mathbf{Y}$ has the following divisor of poles:

$$(\mathbf{Y})_- = \infty_{\mathbf{Y}} + d_{1,\infty}\infty_{\mathbf{X}} + \sum_{\alpha=1}^{H_2}(d_{1,\alpha}+1)q_\alpha + \sum_{\ell=1}^{K_2}\xi_\ell. \tag{2.2}$$

(3) The differential $\mathrm{d}\mathbf{X}$ vanishes (simply) at the (non-marked) points $\{\xi_\ell\}$ and vice versa the differential $\mathrm{d}\mathbf{Y}$ vanishes (simply) at the points $\{\eta_\ell\}$.

All the points entering the above formulae are assumed to be pairwise distinct. The points of the pole divisors which are not marked (the ξ_ℓ, η_ℓ) will be called hard edge. As hinted at in the introduction, these requirements follow from either the loop equations [18] or the exact form of the spectral curve [4]: the points $Q_\alpha := \mathbf{X}(q_\alpha)$ and $X_j := \mathbf{X}(\xi_j)$ are the positions of the poles of the derivatives of the potential $V_1'(X)$ and the hard edges in the $\mathbf{X}$-plane (and conversely for $\mathbf{Y}$): the fact that the ODE for the biorthogonal polynomials has simple poles with nilpotent, rank-1 residue at the points X_j, $j = 1, \ldots,$ implies that the differential $\mathrm{d}\mathbf{X}$ vanishes at one of the points above X_j, at which the eigenvalue $\mathbf{Y}$ has a simple pole.

Under these assumptions we can write the following asymptotic expansions:

$$\mathbf{Y} = \begin{cases} \mathbf{Y} = \sqrt{\dfrac{-2R_j}{\mathbf{X}-X_j}} + \mathcal{O}(1) & \text{near } \xi_j, \quad (\text{here } X_j := \mathbf{X}(\xi_j)) \\ -\displaystyle\sum_{K=0}^{d_{1,\alpha}} \frac{u_{K,\alpha}}{(\mathbf{X}-Q_\alpha)^{K+1}} + \mathcal{O}(1) & \text{near } p_j, \quad (\text{here } Q_\alpha := \mathbf{X}(q_\alpha)) \\ \displaystyle\sum_{K=1}^{d_{1,\infty}+1} u_{K,\infty}\mathbf{X}^{K-1} - \frac{t+\sum_\alpha u_{0,\alpha}}{\mathbf{X}} + \mathcal{O}(\mathbf{X}^{-2}) & \text{near } \infty_{\mathbf{X}} \end{cases}$$

$$\mathbf{X} = \begin{cases} \mathbf{X} = \sqrt{\dfrac{-2S_j}{\mathbf{Y}-Y_j}} + \mathcal{O}(1) & \text{near } \eta_j, \quad (\text{here } Y_j := \mathbf{Y}(\eta_j)) \\ -\displaystyle\sum_{J=0}^{d_{2,\alpha}} \frac{v_{J,\alpha}}{(\mathbf{Y}-P_\alpha)^{J+1}} + \mathcal{O}(1) & \text{near } p_\alpha, \quad (\text{here } P_\alpha := \mathbf{Y}(p_\alpha)) \\ \displaystyle\sum_{J=1}^{d_{2,\infty}+1} v_{J,\infty}\mathbf{Y}^{J-1} - \frac{t+\sum_\alpha v_{0,\alpha}}{\mathbf{Y}} + \mathcal{O}(\mathbf{Y}^{-2}) & \text{near } \infty_{\mathbf{Y}}. \end{cases} \tag{2.3}$$

The above asymptotics imply immediately that there exist two rational functions which we denote by V_1' and V_2', such that

$$\left(\mathbf{Y} - V_1'(\mathbf{X}) + \frac{t}{\mathbf{X}}\right)\mathrm{d}\mathbf{X}, \qquad \left(\mathbf{X} - V_2'(\mathbf{Y}) + \frac{t}{\mathbf{Y}}\right)\mathrm{d}\mathbf{Y} \tag{2.4}$$

are holomorphic differentials in the vicinity of the points $\{\infty_{\mathbf{X}}, q_\alpha, \alpha \geqslant 1\}$ and $\{\infty_{\mathbf{Y}}, p_\alpha, \alpha \geqslant 1\}$, respectively. For later reference, we spell out these functions as

$$V_1(x) := V_{1,\infty}(x) + \sum_\alpha (V_{1,\alpha}(x) - u_{0,\alpha}\ln(x - Q_\alpha))$$
$$V_{1,\infty}(x) := \sum_{K=1}^{d_1+1} \frac{u_{K,\infty}}{K}x^K, \qquad V_{1,\alpha}(x) := \sum_{K=1}^{d_{1,\alpha}} \frac{u_{K,\alpha}}{K(x-Q_\alpha)^K} \tag{2.5}$$

$$V_2(y) := V_{2,\infty}(y) + \sum_\alpha (V_{2,\alpha}(y) - v_{0,\alpha}\ln(y - P_\alpha))$$
$$V_{2,\infty}(y) := \sum_{J=1}^{d_2+1} \frac{v_{J,\infty}}{J}y^J, \qquad V_{2,\alpha}(y) := \sum_{J=1}^{d_{2,\alpha}} \frac{v_{J,\alpha}}{J(y-P_\alpha)^J}. \tag{2.6}$$

A local set of coordinates for the moduli space of these data is provided by the coefficients $\{u_{K,\alpha}, v_{J,\alpha}, t : \alpha = \infty, 1, 2, \ldots\}$, the position of the poles $\{Q_\alpha, P_\alpha\}_{\alpha=1,\ldots}$, the position of the hard-edge divisors $\{X_j, Y_j\}$ together with the so-called filling fractions

$$\epsilon_j := \oint_{a_j} \mathbf{Y}\, \mathrm{d}\mathbf{X}. \tag{2.7}$$

2.1. Bergman kernel

We recall the definition of the Bergman kernel[4] a classical object in complex geometry which can be represented in terms of prime forms and theta functions. In fact we will not need any such sophistication because we are going to use only its fundamental properties (that uniquely determine it).

Let $\{a_i, b_i\}_{i=1\ldots g}$ be a choice of symplectic basis in the homology of the surface Σ_g (which means that a_i intersects only b_i at one point and with positive relative orientation with respect to the natural orientation of the surface Σ_g). The Bergman kernel $\Omega(\zeta, \zeta')$ (where ζ, ζ' denote here and in the following abstract points on the curve) is a bidifferential on $\Sigma_g \times \Sigma_g$ (depending on the fixed choice of homology basis) with the properties

$$\text{Symmetry:} \qquad \Omega(\zeta, \zeta') = \Omega(\zeta', \zeta) \tag{2.8}$$

$$\text{Normalization:} \qquad \oint_{\zeta' \in a_j} \Omega(\zeta, \zeta') = 0 \tag{2.9}$$

$$\oint_{\zeta' \in b_j} \Omega(\zeta, \zeta') = 2\mathrm{i}\pi\omega_j(\zeta) = \text{ the holomorphic normalized Abelian differential.} \tag{2.10}$$

It is holomorphic everywhere on $\Sigma_g \times \Sigma_g \backslash \Delta$, and it has a double pole on the diagonal $\Delta := \{\zeta = \zeta'\}$: namely, if $z(\zeta)$ is any coordinate, we have

$$\Omega(\zeta, \zeta') \underset{\zeta \sim \zeta'}{\simeq} \left[\frac{1}{(z(\zeta) - z(\zeta'))^2} + \frac{1}{6} S_B(\zeta) + \mathcal{O}(z(\zeta) - z(\zeta'))\right] \mathrm{d}z(\zeta)\, \mathrm{d}z(\zeta'), \tag{2.11}$$

where the very important quantity $S_B(\zeta)$ is the 'Bergman projective connection' (it transforms like the Schwartzian derivative under changes of coordinates).

It follows also from the general theory that any normalized Abelian differential of the third kind with simple poles at two points z_- and z_+ with residues, respectively, ± 1 is obtained from the Bergman kernel as

$$\mathrm{d}S_{z_+, z_-}(\zeta) = \int_{\zeta' = z_-}^{z_+} \Omega(\zeta, \zeta'). \tag{2.12}$$

For later purposes, we introduce the dual Bergman kernel defined by

$$\widetilde{\Omega}(\zeta, \zeta') := \Omega(\zeta, \zeta') - 2\pi\mathrm{i} \sum_{j,k=1}^{g} \omega_j(\zeta)\omega_k(\zeta')(\mathbb{B}^{-1})_{jk}, \tag{2.13}$$

where $\mathbb{B}$ is the matrix of b-periods

$$\mathbb{B}_{ij} = \mathbb{B}_{ji} = \oint_{b_j} \omega_i. \tag{2.14}$$

In fact $\widetilde{\Omega}$ is conceptually no different from Ω, being just normalized so that $\oint_{b_j} \widetilde{\Omega} \equiv 0$. We keep the distinction only for later practical purposes.

[4] Our use of the term 'Bergman kernel' is slightly unconventional, since more commonly the Bergman kernel is a reproducing kernel in the L^2 space of holomorphic one-forms. The kernel that we name here 'Bergman' is sometimes referred to as the 'fundamental symmetric bidifferential'. We borrow the (ab)use of the name 'Bergman' from [22].

2.1.1. Prime form. For the sake of completeness, we recall here the definition of the prime form $E(\zeta,\zeta')$.

Definition 2.1. *The prime form $E(\zeta,\zeta')$ is $(-1/2,-1/2)$ bidifferential on $\Sigma_g\times\Sigma_g$*

$$E(\zeta,\zeta')=\frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\mathfrak{u}(\zeta)-\mathfrak{u}(\zeta'))}{h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta)h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta')} \tag{2.15}$$

$$h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta)^2:=\sum_{k=1}^{g}\partial_{\mathfrak{u}_k}\ln\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\bigg|_{\mathfrak{u}=0}\omega_k(\zeta), \tag{2.16}$$

where ω_k are the normalized Abelian holomorphic differentials, $\mathfrak{u}$ is the corresponding Abel map and $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is a half-integer odd characteristic (the prime form does not depend on which one).

Then the relation with the Bergman kernel is as follows:

$$\Omega(\zeta,\zeta')=\mathrm{d}_\zeta\mathrm{d}_{\zeta'}\ln E(\zeta,\zeta')=\sum_{k,j=1}^{g}\partial_{\mathfrak{u}_k}\partial_{\mathfrak{u}_j}\ln\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\bigg|_{\mathfrak{u}(\zeta)-\mathfrak{u}(\zeta')}\omega_k(\zeta)\omega_j(\zeta'). \tag{2.17}$$

Remark 2.1. In genus zero, of course, there are no theta functions: however, there is a Bergman kernel with the same properties, given simply by (using the standard coordinate on the complex plane)

$$\Omega(z,z')=\frac{\mathrm{d}z\,\mathrm{d}z'}{(z-z')^2}. \tag{2.18}$$

3. Planar limit of the free energy

The planar limit of the free energy is defined by the following set of compatible equations:

$$\begin{array}{l|l}
\partial_{u_{K,0}}\mathcal{F}=U_{K,0}:=-\frac{1}{K}\operatorname*{res}_{\infty_{\mathbf{X}}}\mathbf{X}^K\mathbf{Y}\,\mathrm{d}\mathbf{X} & \partial_{v_{J,0}}\mathcal{F}=V_{J,0}:=-\frac{1}{J}\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{Y}^J\mathbf{X}\,\mathrm{d}\mathbf{Y}\\
\partial_{u_{K,\alpha}}\mathcal{F}=U_{K,\alpha}:=-\frac{1}{K}\operatorname*{res}_{q_\alpha}\frac{1}{(\mathbf{X}-Q_\alpha)^K}\mathbf{Y}\,\mathrm{d}\mathbf{X} & \partial_{v_{J,\alpha}}\mathcal{F}=V_{K,\alpha}:=-\frac{1}{J}\operatorname*{res}_{p_\alpha}\frac{1}{(\mathbf{Y}-P_\alpha)^J}\mathbf{X}\,\mathrm{d}\mathbf{Y}\\
\partial_{u_{0,\alpha}}\mathcal{F}=U_{0,\alpha}:=⨍_{q_\alpha}^{\infty_{\mathbf{X}}}\mathbf{Y}\,\mathrm{d}\mathbf{X} & \partial_{v_{0,\alpha}}\mathcal{F}=V_{0,\alpha}:=⨍_{p_\alpha}^{\infty_{\mathbf{Y}}}\mathbf{X}\,\mathrm{d}\mathbf{Y}\\
\partial_{X_j}\mathcal{F}=R_j:=\frac{1}{2}\operatorname*{res}_{\xi_j}\mathbf{Y}^2\,\mathrm{d}\mathbf{X} & \partial_{Y_j}\mathcal{F}=S_j:=\frac{1}{2}\operatorname*{res}_{\eta_j}\mathbf{X}^2\,\mathrm{d}\mathbf{Y}\\
\partial_{Q_\alpha}\mathcal{F}=\operatorname*{res}_{q_\alpha}\left(V'_{1,\alpha}(\mathbf{X})-\frac{u_{0,\alpha}}{(\mathbf{X}-Q_\alpha)}\right)\mathbf{Y}\,\mathrm{d}\mathbf{X} & \partial_{P_\alpha}\mathcal{F}=\operatorname*{res}_{p_\alpha}\left(V'_{2,\alpha}(\mathbf{Y})-\frac{v_{0,\alpha}}{(\mathbf{Y}-P_\alpha)}\right)\mathbf{X}\,\mathrm{d}\mathbf{Y}
\end{array}$$

$$[-1pt]\partial_t\mathcal{F}=\mu:=⨍_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\mathbf{Y}\,\mathrm{d}\mathbf{X}-\sum_{\alpha\geqslant1}v_{0,\alpha}=⨍_{\infty_{\mathbf{X}}}^{\infty_{\mathbf{Y}}}\mathbf{X}\,\mathrm{d}\mathbf{Y}-\sum_{\alpha\geqslant1}u_{0,\alpha}$$

$$\partial_{\epsilon_j}\mathcal{F}=\Gamma_j:=\frac{1}{2\mathrm{i}\pi}\oint_{b_j}\mathbf{Y}\,\mathrm{d}\mathbf{X}. \tag{3.1}$$

In these formulae the symbol $⨍$ stands for the regularized integral obtained by subtraction of the singular part in the local parameter as follows:

(i) at $\infty_{\mathbf{X}}(\infty_{\mathbf{Y}})$ the local parameter is $z=\mathbf{X}^{-1}(\widetilde{z}=\mathbf{Y}^{-1})$;
(ii) at $q_\alpha(p_\alpha)$ the local parameter is $z_\alpha=\mathbf{X}-Q_\alpha$ $(z_{\widetilde{\alpha}}=\mathbf{Y}-P_\alpha)$.

The regularization is then defined as follows: if z is any of the above local parameters then

$$\fint^{0} \omega := \lim_{\epsilon \to 0} \int^{\epsilon} \omega - f(\epsilon), \tag{3.2}$$

where $f(z)$ is defined as the antiderivative (without constant) of the singular part of $\frac{\omega}{\mathrm{d}z}$ as a function of z (near $z = 0$).

Example 3.1. The regularized integral according to the definition is

$$\fint_{\infty_{\mathbf{X}}}^{q_\alpha} \mathbf{Y}\,\mathrm{d}\mathbf{X} := \lim_{\epsilon \to q_\alpha} \lim_{R \to \infty_{\mathbf{X}}} \int_{R}^{\epsilon} \mathbf{Y}\,\mathrm{d}\mathbf{X} + V_{1,\infty}(\mathbf{X}(R)) - \left(t - \sum_\alpha u_{0,\alpha}\right) \ln(\mathbf{X}(R)) - V_{1,\alpha}(\mathbf{X}(\epsilon)). \tag{3.3}$$

The two expressions for μ in (3.1) are proven to be equivalent (thus showing the symmetry in the roles of $\mathbf{X}$ and $\mathbf{Y}$) by integration by parts, paying attention at the definition of the regularization (which involves as local parameters $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$ at the two different poles); indeed we have

$$\fint_{p}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} = \lim_{\epsilon \to \infty_{\mathbf{X}}} \left(\int_{p}^{\epsilon} \mathbf{Y}\,\mathrm{d}\mathbf{X} - V_{1,\infty}(\mathbf{X}(\epsilon)) + \left(t + \sum_\alpha u_{0,\alpha}\right) \ln \mathbf{X}(\epsilon) \right) \tag{3.4}$$

$$= \lim_{\epsilon \to \infty_{\mathbf{X}}} \left(- \int_{p}^{\epsilon} \mathbf{X}\,\mathrm{d}\mathbf{Y} + \mathbf{X}(\epsilon) \overbrace{\mathbf{Y}(\epsilon)}^{V'_{1\infty}(\mathbf{X}) - \mathbf{X}^{-1}(t + \sum u_{0,\alpha}) + \dots} - \mathbf{X}(p)\mathbf{Y}(p) + V_{1,\infty}(\mathbf{X}(\epsilon)) \right.$$
$$\left. + \left(t + \sum_\alpha u_{0,\alpha}\right) \ln \mathbf{X}(\epsilon) \right) \tag{3.5}$$

$$= - \fint_{p}^{\infty_{\mathbf{X}}} \mathbf{X}\,\mathrm{d}\mathbf{Y} - \mathbf{X}(p)\mathbf{Y}(p) - \left(t + \sum_\alpha u_{0,\alpha}\right) \tag{3.6}$$

together with a similar formula for the symmetric expression

$$\fint_{\infty_{\mathbf{Y}}}^{p} \mathbf{Y}\,\mathrm{d}\mathbf{X} = \mathbf{X}(p)\mathbf{Y}(p) + \left(t + \sum_\beta v_{0,\beta}\right) - \fint_{\infty_{\mathbf{Y}}}^{p} \mathbf{X}\,\mathrm{d}\mathbf{Y}. \tag{3.7}$$

Combining the two, one has

$$\mu = \fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} - \sum_\beta v_{0,\beta} = \fint_{\infty_{\mathbf{X}}}^{\infty_{\mathbf{Y}}} \mathbf{X}\,\mathrm{d}\mathbf{Y} - \sum_\alpha u_{0,\alpha}. \tag{3.8}$$

In full generality, given any meromorphic differential and local parameters around its poles, one can give completely explicit formulae for its regularized integrals (see appendix D). In our specific setting we give explicit formulae of the previous regularized integrals in terms of canonical differentials of the third kind in appendix A.

We also make the important remark that in order for the above formulae to make sense, we must perform some surgery on the surface by cutting it along a choice of the a, b-cycles and by performing some mutually non-intersecting cuts between the poles with nonzero residues of the differential $\mathbf{Y}\,\mathrm{d}\mathbf{X}$ (see figure 1). We achieve this goal by choosing some segments on the surface joining the chosen basepoint for the canonical dissection to $\infty_{\mathbf{X}}$ and $\infty_{\mathbf{Y}}$, and

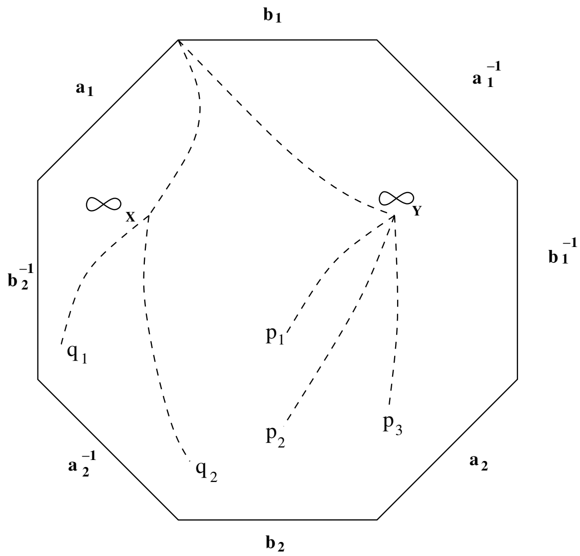


Figure 1. A visualization of an example of the dissection mentioned in the text for a genus-2 curve.

then segments connecting $\infty_{\mathbf{X}}$ to q_α and $\infty_{\mathbf{Y}}$ to p_α. The result of this dissection is a simply-connected domain where $\mathbf{X}, \mathbf{Y}$ are meromorphic functions and where the regularizations involving logarithms are defined by taking the principal determination.

The compatibility of equations (3.1) for $\mathcal{F}$ can be shown by taking the cross-derivatives. We now briefly recall, for the reader's sake, how to compute them since much of the formalism is needed in the following. The main tool is the previously defined Bergman kernel (section 2.1) providing an effective way of writing formulae for first-, second- and third-kind normalized differentials on the Riemann surface. This is needed when computing the cross derivatives of the free energy since the differentials $\partial \mathbf{Y}\,\mathrm{d}\mathbf{X}$ and $\partial \mathbf{X}\,\mathrm{d}\mathbf{Y}$ (here ∂ is any variation of the coordinates) can be identified with certain canonical differentials.

Let us first recall the *thermodynamic identity*

$$(\partial \mathbf{Y})_{\mathbf{X}}\,\mathrm{d}\mathbf{X} = -(\partial \mathbf{X})_{\mathbf{Y}}\,\mathrm{d}\mathbf{Y}, \tag{3.9}$$

where the subscript denotes the local coordinate to be kept fixed under variation. As an example of the use of (3.9) in identifying the various differentials, we consider a derivative ∂_{u_K}. From the defining relations for the coordinates (2.3), we see that

$$\left(\partial_{u_{K,\infty}} \mathbf{Y}\right)_{\mathbf{X}} \mathrm{d}\mathbf{X} = \begin{cases} \mathbf{X}^{K-1}\,\mathrm{d}\mathbf{X} + \mathcal{O}(\mathbf{X}^{-2})\,\mathrm{d}\mathbf{X} & \text{near } \infty_{\mathbf{X}} \\ \mathcal{O}(1)\,\mathrm{d}\mathbf{X} & \text{near } q_\alpha \end{cases} \tag{3.10}$$

has a pole of order K at $\infty_{\mathbf{X}}$ without residue. In order what kind of singularity it has at $\infty_{\mathbf{Y}}$, we use (3.9) followed by (2.3)

$$\left(\partial_{u_{K,\infty}} \mathbf{Y}\right)_{\mathbf{X}} \mathrm{d}\mathbf{X} = -\left(\partial_{u_K} \mathbf{X}\right)_{\mathbf{Y}} \mathrm{d}\mathbf{Y} = \begin{cases} \mathcal{O}(\mathbf{Y}^{-2})\,\mathrm{d}\mathbf{Y} & \text{near } \infty_{\mathbf{Y}} \\ \mathcal{O}(1)\,\mathrm{d}\mathbf{Y} & \text{near } p_\alpha. \end{cases} \tag{3.11}$$

Therefore, the differential $\left(\partial_{u_{K,\infty}} \mathbf{Y}\right)_{\mathbf{X}} \mathrm{d}\mathbf{X}$ has only a pole at $\infty_{\mathbf{X}}$ and no residues: moreover, it follows by differentiation of (2.7) that this differential is also normalized (i.e., with vanishing a-cycles), which is sufficient to uniquely specify it. It is then an exercise using the properties of Ω to see that

$$\left(\partial_{u_{K,0}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = -\operatorname*{res}_{\infty_{\mathbf{X}}}\frac{\mathbf{X}^K}{K}\Omega. \tag{3.12}$$

Following the same logic and similar reasoning, one can prove the following formulae:

$$\text{First kind}\ \left(\partial_{\epsilon_j}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \omega_j = \frac{1}{2\mathrm{i}\pi}\oint_{b_j}\Omega \tag{3.13}$$

$$\text{Second kind}\begin{cases}\left(\partial_{u_{K,\infty}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = -\operatorname*{res}_{\infty_{\mathbf{X}}}\dfrac{\mathbf{X}^K}{K}\Omega =: \omega_{K,\infty}\\ \left(\partial_{u_{K,\alpha}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = -\dfrac{1}{K}\operatorname*{res}_{q_\alpha}(\mathbf{X}-Q_\alpha)^{-K}\Omega =: \omega_{K,\alpha}\\ \left(\partial_{X_j}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \operatorname*{res}_{\xi_j}\mathbf{Y}\Omega =: \omega_{X_j}\\ \left(\partial_{Q_\alpha}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \operatorname*{res}_{q_\alpha}\left(V'_{1,\alpha}(\mathbf{X}) - \dfrac{u_{0,\alpha}}{(\mathbf{X}-Q_\alpha)}\right)\Omega\\ \left(\partial_{v_{J,\infty}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \operatorname*{res}_{\infty_{\mathbf{Y}}}\dfrac{\mathbf{Y}^J}{J}\Omega =: \omega_{\tilde{J},\infty}\\ \left(\partial_{v_{J,\alpha}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \dfrac{1}{J}\operatorname*{res}_{p_\alpha}(\mathbf{Y}-P_\alpha)^{-J}\Omega = \omega_{\tilde{J},\alpha}\\ \left(\partial_{Y_j}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{Y} = -\operatorname*{res}_{\eta_j}\mathbf{X}\Omega =: \omega_{Y_j}\\ \left(\partial_{P_\alpha}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = -\operatorname*{res}_{p_\alpha}\left(V'_{2,\alpha}(\mathbf{Y}) - \dfrac{v_{0,\alpha}}{(\mathbf{Y}-P_\alpha)}\right)\Omega\end{cases} \tag{3.14}$$

$$\text{Third kind}\begin{cases}\left(\partial_{u_{0,\alpha}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \displaystyle\int_{q_\alpha}^{\infty_{\mathbf{X}}}\Omega =: \omega_{0,\alpha}\\ \left(\partial_{v_{0,\alpha}}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = -\displaystyle\int_{p_\alpha}^{\infty_{\mathbf{Y}}}\Omega =: \omega_{\tilde{0},\alpha}\\ (\partial_t\mathbf{Y})_{\mathbf{X}}\mathrm{d}\mathbf{X} = \displaystyle\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\Omega =: \omega_0.\end{cases} \tag{3.15}$$

The only formulae above that need some further explanations are those for the derivatives with respect to X_j (or similarly Y_j); from the asymptotic behaviour (2.3) in the local parameter $z = \sqrt{\mathbf{X}-X_j}$, we have

$$\begin{aligned}\partial_{X_j}\mathbf{Y}\,\mathrm{d}\mathbf{X} &= \left[\frac{\left(\partial_{X_j}X_j\right)}{2}\frac{\sqrt{-2R_j}}{z^3} + \frac{\partial\sqrt{-2R_j}}{z} + \mathcal{O}(1)\right]2z\,\mathrm{d}z\\ &= \frac{\sqrt{-2R_j}}{z^2}\mathrm{d}z + \mathcal{O}(1) = -\mathrm{d}\left(\frac{\sqrt{-2R_j}}{z} + \mathcal{O}(1)\right) = \operatorname*{res}_{\xi_j}\mathbf{Y}\Omega.\end{aligned} \tag{3.16}$$

This proves that if $\partial = \partial_{X_j}$ then the differential has a double pole at ξ_j without residues: similar reasoning at the other singularities and for the a-cycles of the differential force it to be equal to the above formula in (3.14).

Remark 3.1. As explained in the introduction, we are also interested to the restriction of $\mathcal{F}$ to the subvariety of the moduli space defined by

$$\partial_{\epsilon_j}\mathcal{F}(V_1, V_2, t, \underline{\epsilon}(V_1, V_2, t)) \equiv 0, \qquad j = 1, \ldots, g. \tag{3.17}$$

Since on this subvariety the differentials $\mathbf{X}\,\mathrm{d}\mathbf{Y}$, $\mathbf{Y}\,\mathrm{d}\mathbf{X}$ have identically vanishing b-periods, the formulae for the constrained derivatives that substitute (3.14), (3.15)[5] are the same with $\widetilde{\Omega}$ (2.13) replacing Ω.

Before writing the cross derivatives in a way which is symmetric in $\mathbf{X}, \mathbf{Y}$, we introduce some useful notation: all the differentials (3.13)–(3.15) are obtained by applying a suitable integral operator to one variable of the Bergman kernel Ω according to the following table of translation:

$$\begin{array}{l|l}
\dfrac{\partial}{\partial u_{K,\infty}} \mapsto \mathcal{U}_{K,\infty} := -\dfrac{1}{2\mathrm{i}K\pi}\displaystyle\oint_{\infty_{\mathbf{X}}} \mathbf{X}^K & \dfrac{\partial}{\partial v_{J,\infty}} \mapsto \mathcal{V}_{J,\infty} := \dfrac{1}{2\mathrm{i}J\pi}\displaystyle\oint_{\infty_{\mathbf{Y}}} \mathbf{Y}^J \\
\dfrac{\partial}{\partial u_{0,\alpha}} \mapsto \mathcal{U}_{0,\alpha} := \displaystyle -\!\!\!\!\!\!\int_{q_\alpha}^{\infty_{\mathbf{X}}} & \dfrac{\partial}{\partial v_{0,\alpha}} \mapsto \mathcal{V}_{0,\alpha} := \displaystyle -\!\!\!\!\!\!\int_{p_\alpha}^{\infty_{\mathbf{Y}}} \\
\dfrac{\partial}{\partial u_{K,\alpha}} \mapsto \mathcal{U}_{K,\alpha} := -\dfrac{1}{2\mathrm{i}K\pi}\displaystyle\oint_{q_\alpha} \dfrac{1}{(\mathbf{X}-Q_\alpha)^K} & \dfrac{\partial}{\partial v_{J,\alpha}} \mapsto \mathcal{V}_{J,\alpha} := \dfrac{1}{2\mathrm{i}J\pi}\displaystyle\oint_{p_\alpha} \dfrac{1}{(\mathbf{Y}-P_\alpha)^J} \\
\dfrac{\partial}{\partial X_j} \mapsto \mathcal{R}_J := \dfrac{1}{2\mathrm{i}\pi}\displaystyle\oint_{\xi_j} \mathbf{Y} & \dfrac{\partial}{\partial Y_j} \mapsto \mathcal{S}_J := -\dfrac{1}{2\mathrm{i}\pi}\displaystyle\oint_{\eta_j} \mathbf{X}
\end{array}$$

$$\frac{\partial}{\partial t} \mapsto \mathcal{T} := -\!\!\!\!\!\!\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}$$

$$\frac{\partial}{\partial \epsilon_j} \mapsto \mathcal{E}_j := \frac{1}{2\mathrm{i}\pi}\oint_{b_j}. \tag{3.18}$$

All the differentials (3.13)–(3.15) are obtained by applying the corresponding integral operator in (3.18) to the Bergman bidifferential Ω.

In order to write the cross derivatives, let us choose two coordinates and denote by ∂_1, ∂_2 the corresponding derivatives and by $\int_{\partial_1}, \int_{\partial_2}$ the corresponding integral operator as per table (3.18): then we have

$$\partial_1\partial_2\mathcal{F} = \partial_1 \int_{\partial_2} \mathbf{Y}\,\mathrm{d}\mathbf{X} = \int_{\partial_2} (\partial_1\mathbf{Y})_{\mathbf{X}}\,\mathrm{d}\mathbf{X} = \int_{\partial_2}\int_{\partial_1} \Omega. \tag{3.19}$$

The important and conclusive remark now is that the order of the action of the integral operators appearing in the list (3.18) on Ω is immaterial because the kernel Ω is symmetric and, more importantly, because its residue on the diagonal is zero. This means that in exchanging two integral operators one may in fact acquire the integral of a total differential which is going to cancel either by integration or against the regularization. To illustrate the point we make two examples.

Example 3.2. Consider two coordinates $u_{K,\alpha}, v_{J,\beta}$: then

$$\partial_{u_{K,\alpha}}\partial_{v_{J,\beta}}\mathcal{F} = \mathcal{V}_{J,\beta}\mathcal{U}_{K,\alpha}\Omega. \tag{3.20}$$

In this case the two integral operators involve either residues (for $K > 0$) or (regularized) integrals. Either way the contours do not intersect and the double integral is independent of the order.

Example 3.3. Consider the derivatives $\partial_{u_{0,\alpha}}$ and $\partial_{u_{K,\alpha}}$; in this case the integral operators do involve intersecting contours, hence care must be exercised

$$\partial_{u_{0,\alpha}}\partial_{u_{K,\alpha}}\mathcal{F} = \frac{1}{2\mathrm{i}K\pi}\oint_{q_\alpha} (\mathbf{X}-Q_\alpha)^{-K}(\zeta) -\!\!\!\!\!\!\int_{\infty_{\mathbf{X}}}^{q_\alpha} \Omega(\zeta,\xi). \tag{3.21}$$

[5] Equations (3.13) do not make sense on the subvariety since ϵ are not independent coordinates any longer.

The inner integral in fact does not need any regularization, so we have

$$\partial_{u_{0,\alpha}}\partial_{u_{K,\alpha}}\mathcal{F} = \frac{1}{2\mathrm{i}K\pi}\oint_{q_\alpha}(\mathbf{X}-Q_\alpha)^{-K}(\zeta)\int_{\infty_\mathbf{X}}^{q_\alpha}\Omega(\zeta,\xi) \tag{3.22}$$

$$= \lim_{\epsilon\to q_\alpha}\frac{1}{2\mathrm{i}K\pi}\int_{\infty_\mathbf{X}}^{\epsilon}\oint_{q_\alpha}(\mathbf{X}-Q_\alpha)^{-K}(\zeta)\Omega(\zeta,\xi) - \frac{1}{K}(\mathbf{X}(\epsilon)-Q_\alpha)^{-K} \tag{3.23}$$

$$= \frac{1}{2\mathrm{i}K\pi}\,⨍_{\infty_\mathbf{X}}^{q_\alpha}\oint_{q_\alpha}(\mathbf{X}-Q_\alpha)^{-K}(\zeta)\Omega(\zeta,\xi) = \partial_{u_{0,\alpha}}\partial_{u_{K,\alpha}}\mathcal{F}. \tag{3.24}$$

(The exchange of the order of the integrals gives a $-2\mathrm{i}\pi\delta$ supported at the intersection of the contours of integration.)

Theorem 3.1. *The free energy is given by the formula (we set $u_{K,0} := u_{K,\infty}$, $v_{J,0} := v_{J,\infty}$, $u_{0,0} = u_{0,\infty} := v_{0,0} = v_{0,\infty} := 0$ for uniformity in the formulae)*

$$2\mathcal{F} = \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}U_{K,\alpha} + \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}V_{J,\alpha} + t\mu + \sum_{j=1}^{g}\epsilon_j\Gamma_j + \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\zeta\in\mathcal{D}_\mathbf{X}}\operatorname*{res}_{\zeta}\mathbf{Y}^2\mathbf{X}\,\mathrm{d}\mathbf{X} \\ \dfrac{1}{2}\displaystyle\sum_{\zeta\in\mathcal{D}_\mathbf{Y}}\operatorname*{res}_{\zeta}\mathbf{X}^2\mathbf{Y}\,\mathrm{d}\mathbf{Y}, \end{cases} \tag{3.25}$$

where

$$\mathcal{D}_\mathbf{X} := \{\infty_\mathbf{X}, q_\alpha, \xi_j, \alpha = 1,\ldots; j = 1,\ldots\} \tag{3.26}$$

$$\mathcal{D}_\mathbf{Y} := \{\infty_\mathbf{Y}, p_\alpha, \eta_j, \alpha = 1,\ldots; j = 1,\ldots\} \tag{3.27}$$

(see definitions of the properties of the points appearing here at the beginning of section 2)[6].

Proof. First of all note that the expression is symmetric in the roles of $\mathbf{X}$, $\mathbf{Y}$ after integration by parts and moving the residues to the other poles

$$\frac{1}{2}\sum_{\zeta\in\mathcal{D}_\mathbf{X}}\operatorname*{res}_{\zeta}\mathbf{Y}^2\mathbf{X}\,\mathrm{d}\mathbf{X} = -\frac{1}{2}\sum_{\zeta\in\mathcal{D}_\mathbf{Y}}\operatorname*{res}_{\zeta}\mathbf{Y}^2\mathbf{X}\,\mathrm{d}\mathbf{X} = \frac{1}{2}\sum_{\zeta\in\mathcal{D}_\mathbf{Y}}\operatorname*{res}_{\zeta}\mathbf{X}^2\mathbf{Y}\,\mathrm{d}\mathbf{Y}, \tag{3.28}$$

where we have used that $\mathcal{D}_\mathbf{X}\cup\mathcal{D}_\mathbf{Y}$ is the set of all poles of the differential $\mathbf{Y}^2\mathbf{X}\,\mathrm{d}\mathbf{X}$. Now, the proposed expression is nothing, but

$$2\mathcal{F} = \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}(\mathbf{Y}\,\mathrm{d}\mathbf{X}) - \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}(\mathbf{X}\,\mathrm{d}\mathbf{Y}) + t\mathcal{T}(\mathbf{Y}\,\mathrm{d}\mathbf{X}) \tag{3.29}$$

$$+\sum_{j=1}^{g}\epsilon_j\mathcal{E}_j(\mathbf{Y}\,\mathrm{d}\mathbf{X}) + \frac{1}{2}\sum_{\zeta\in\mathcal{D}_\mathbf{X}}\operatorname*{res}_{\zeta}\mathbf{Y}^2\mathbf{X}\,\mathrm{d}\mathbf{X} - t\sum v_{0,\alpha}. \tag{3.30}$$

Suppose we compute a derivative with respect to $u_{R,\beta}$: using the list of differentials (3.13)–(3.15) and moving the computation of residues over to $\mathcal{D}_\mathbf{Y}$, for convenience, before the differentiation, we have

$$2\partial_{u_{R,\beta}}\mathcal{F} = \overbrace{\mathcal{U}_{R,\beta}(\mathbf{Y}\,\mathrm{d}\mathbf{X})}^{=U_{R,\beta}} + \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}(\mathcal{U}_{R,\beta}\Omega) \tag{3.31}$$

[6] The set $\mathcal{D}_\mathbf{X}$ is the support of the pole-divisor of $\mathbf{Y}$ less the point $\infty_\mathbf{Y}$, and vice versa for $\mathcal{D}_\mathbf{Y}$.

$$
-\sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}(-\mathcal{U}_{R,\beta}\Omega) + t\mathcal{T}(\mathcal{U}_{R,\beta}\Omega) + \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j(\mathcal{U}_{R,\beta}\Omega) - \sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{Y}\mathbf{X}\mathcal{U}_{R,\beta}(\Omega) \tag{3.32}
$$

$$
= U_{R,\beta} + \mathcal{U}_{R,\beta}\left(\sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}(\Omega) + \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}(\Omega)\right.
$$

$$
\left. + t\mathcal{T}(\Omega) + \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j(\Omega) - \sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{X}\mathbf{Y}\Omega\right). \tag{3.33}
$$

Note that the operator $\mathcal{U}_{R,\beta}$ involves residues at one of the points of $\mathcal{D}_{\mathbf{X}}$ and hence commutes with the other residues when acting on the (singular) kernel Ω also for the last term involving residues at $\mathcal{D}_{\mathbf{Y}}$.

From the properties of Ω and the definitions of the integral operators, it follows that the differential acted upon by $\mathcal{U}_{R,\beta}$ is precisely $\mathbf{Y}\,\mathrm{d}\mathbf{X}$, namely

$$
\mathbf{Y}\,\mathrm{d}\mathbf{X} = \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}(\Omega) + \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}(\Omega)
$$

$$
+ t\mathcal{T}(\Omega) + \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j(\Omega) - \sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{X}\mathbf{Y}\Omega. \tag{3.34}
$$

This can be seen by analysing the singular behaviour near the poles and the a-periods of both sides of the equality and verifying that they are the same[7]. Whence we have the desired conclusion of this part of the proof. The other derivatives are treated in completely parallel way.

The derivatives with respect to X_j, Y_j are a little different because there is no explicit dependence of $\mathcal{F}$ from these coordinates. However, this produces the correct result since, for example

$$
2\partial_{X_\ell}\mathcal{F} = \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}\left(\mathcal{R}_\ell\Omega\right) - \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}\left(-\mathcal{R}_\ell\Omega\right) + t\mathcal{T}\left(\mathcal{R}_\ell\Omega\right) \tag{3.35}
$$

$$
+ \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j\left(\mathcal{R}_\ell\Omega\right) - \sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{X}\mathbf{Y}\mathcal{R}_\ell(\Omega) \tag{3.36}
$$

$$
= \mathcal{R}_l(\mathbf{Y}\,\mathrm{d}\mathbf{X}), \tag{3.37}
$$

which is consistent with our definitions (3.1).

As a final case, we compute the derivative with respect to Q_α: here some care should be paid to the commutation of the derivative with the integral operators. Indeed ∂_{Q_α} does not commute with the integral operators $\mathcal{U}_{K,\alpha}$, $K = 0, \ldots$, but instead we have

$$
\left[\partial_{Q_\alpha}, \mathcal{U}_{K,\alpha}\right] = K\mathcal{U}_{K+1,\alpha}, \qquad K = 1, \ldots \tag{3.38}
$$

$$
\left[\partial_{Q_\alpha}, \mathcal{U}_{0,\alpha}\right] = \mathcal{U}_{1,\alpha}. \tag{3.39}
$$

[7] One should use that the behaviour near a pole of the last term on the lhs is, e.g.

$$
\sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{X}\mathbf{Y}\Omega \underset{p_\alpha}{\sim} -\mathrm{d}\left(\mathbf{Y}V'_{2,\alpha}(\mathbf{Y})\right).
$$

While (3.38) is rather obvious from the definition of the integral operator, some explanation is necessary for (3.39). Expanding $\fint_{\epsilon}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X}$ in the local parameter at q_α, we have

$$\fint_{\epsilon}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} = -V_{1,\alpha}(\mathbf{X}(\epsilon)) + c_0 + c_1 z_\alpha + \mathcal{O}\big(z_\alpha^2\big) + V_{1,\alpha}(\mathbf{X}(\epsilon)). \tag{3.40}$$

Therefore, we have

$$\partial_{Q_\alpha} \fint_{q_\alpha}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} = \partial_{Q_\alpha}\left(\lim_{\epsilon\to q_\alpha} \int_{\epsilon} \mathbf{Y}\,\mathrm{d}\mathbf{X} + V_{1,\alpha}(\mathbf{X}(\epsilon))\right) = \partial_{Q_\alpha} c_0. \tag{3.41}$$

Vice versa (recalling that $\partial_{Q_\alpha} z_\alpha = -1$)

$$\begin{aligned}\fint_{q_\alpha}^{\infty_{\mathbf{X}}} \big(\partial_{Q_\alpha}\mathbf{Y}\big)_{\mathbf{X}}\,\mathrm{d}\mathbf{X} &= \lim_{\epsilon\to q_\alpha}\int_{\epsilon}^{\infty_{\mathbf{X}}} \big(\partial_{Q_\alpha}\mathbf{Y}\big)_{\mathbf{X}}\,\mathrm{d}\mathbf{X} \\ &= \lim_{\epsilon\to q_\alpha}\big(\partial_{Q_\alpha} c_0 - c_1 + \partial_{Q_\alpha} c_1 z_\alpha + \mathcal{O}\big(z_\alpha^2\big)\big) = \partial_{Q_\alpha} c_0 - c_1.\end{aligned} \tag{3.42}$$

This shows that

$$\left[\partial_{Q_\alpha}, \fint_{q_\alpha}\right] = -\frac{1}{2\mathrm{i}\pi}\oint_{q_\alpha}\frac{1}{\mathbf{X}-Q_\alpha} = \mathcal{U}_{1,\alpha}. \tag{3.43}$$

Using this and computing the derivative of $\mathcal{F}$, we obtain the desired result

$$\partial_{Q_\alpha}\mathcal{F} = \operatorname*{res}_{q_\alpha} V'_{1,\alpha}(\mathbf{X})\mathbf{Y}\,\mathrm{d}\mathbf{X}. \tag{3.44}$$

Finally, while the reasoning is mostly similar, the t derivative has an additional technical difficulty. First of all we have

$$\partial_t \fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} = \fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}} \Omega + 1. \tag{3.45}$$

The reason of the additional +1 is the fact that the local parameters near the two poles are different functions (here we set for brevity $t_{\mathbf{X}} = t + \sum u_{0,\alpha}$, $t_{\mathbf{Y}} = t + \sum v_{0,\alpha}$)

$$\begin{aligned}\partial_t \fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} &= \lim_{\substack{\epsilon\to\infty_{\mathbf{Y}}\\ \rho\to\infty_{\mathbf{X}}}} \partial_t\bigg[\int_{\epsilon}^{\rho} \mathbf{Y}\,\mathrm{d}\mathbf{X} - (V_{1,\infty}(\mathbf{X}) - t_{\mathbf{X}}\ln(\mathbf{X}))_{\rho} \\ &\qquad + (\mathbf{Y}V'_{2,\infty}(\mathbf{Y}) - V_{2,\infty}(\mathbf{Y}) - t_{\mathbf{Y}}\ln(\mathbf{Y}))_{\epsilon}\bigg] \\ &= \lim_{\substack{\epsilon\to\infty_{\mathbf{Y}}\\ \rho\to\infty_{\mathbf{X}}}}\left[\int_{\epsilon}^{\rho}\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\Omega + \ln(\mathbf{X}(\rho)) - \ln(\mathbf{Y}(\epsilon)) + \left(\mathbf{Y}V''_{2,\infty}(\mathbf{Y}) - \frac{t_{\mathbf{Y}}}{\mathbf{Y}}\right)\overset{\frac{\mathrm{d}\mathbf{Y}}{\mathbf{Y}\,\mathrm{d}\mathbf{X}}+\cdots}{\overset{\|}{(\partial_t\mathbf{Y})_{\mathbf{X}}}}\right] \\ &= \fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\Omega + 1.\end{aligned} \tag{3.46}$$

Moreover, whether we sum at the poles in $\mathcal{D}_{\mathbf{X}}$ or $\mathcal{D}_{\mathbf{Y}}$, we need to interchange the order of the following residue/integral

$$\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{X}\mathbf{Y}\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\Omega = \lim_{\epsilon\to\infty_{\mathbf{Y}}}\int_{\epsilon}^{\infty_{\mathbf{X}}}\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{X}\mathbf{Y}\Omega - \mathbf{Y}(\epsilon)\overset{V'_{2,\infty}(\mathbf{Y})-(t+\sum v_{0,\alpha})\mathbf{Y}^{-1}+\cdots}{\overset{\|}{\mathbf{X}(\epsilon)}} = \fint_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{X}\mathbf{Y}\Omega + t + \sum_\alpha v_{0,\alpha}. \tag{3.47}$$

Putting it all together, we find

$$2\partial_t\mathcal{F} = \left(\mathcal{T}(\mathbf{Y}\,\mathrm{d}\mathbf{X}) + t - \sum_\alpha v_{0,\alpha}\right) + \mathcal{T}(\mathbf{Y}\,\mathrm{d}\mathbf{X}) - t - \sum_\alpha v_{0,\alpha} = 2\mu. \tag{3.48}$$

The other derivatives with respect to the moduli $v_{J,\alpha}$, Y_j, P_α are computed in a similar way by first rewriting the expression for $\mathcal{F}$ equivalently in the symmetric way with respect to the exchange of roles of $\mathbf{X}$, $\mathbf{Y}$. □

Corollary 3.1. *The free energy satisfies the following scaling constraints:*

$$2\mathcal{F} = \mathbb{V}_{\mathbf{Y}}\mathcal{F} + \sum_{1\leqslant\alpha<\beta} v_{0,\alpha}v_{0,\beta} + t\sum_{\alpha\geqslant 1} v_{0,\alpha} + \frac{t^2}{2} \tag{3.49}$$

$$2\mathcal{F} = \mathbb{V}_{\mathbf{X}}\mathcal{F} + \sum_{1\leqslant\alpha<\beta} u_{0,\alpha}u_{0,\beta} + t\sum_{\alpha\geqslant 1} u_{0,\alpha} + \frac{t^2}{2} \tag{3.50}$$

where

$$\begin{aligned}\mathbb{V}_{\mathbf{Y}} &:= \sum_{\alpha\geqslant 0}\sum_{K\geqslant 0} u_{K,\alpha}\frac{\partial}{\partial u_{K,\alpha}} + \sum_{J=1}^{d_{2,\infty}}(1-J)v_{J,\infty}\frac{\partial}{\partial v_{J,\infty}} + \sum_{\alpha\geqslant 1}\left(P_\alpha\frac{\partial}{\partial P_\alpha} + \sum_{J=0}^{d_{2,\alpha}}(J+1)v_{J,\alpha}\frac{\partial}{\partial v_{J,\alpha}}\right)\\ &\quad + \sum_j Y_j\frac{\partial}{\partial Y_j} + t\frac{\partial}{\partial t} + \sum_{j=1}^{g}\epsilon_j\frac{\partial}{\partial \epsilon_j}\\ \mathbb{V}_{\mathbf{X}} &:= \sum_{\alpha\geqslant 0}\sum_{J\geqslant 0} v_{J,\alpha}\frac{\partial}{\partial v_{J,\alpha}} + \sum_{K=1}^{d_{1,\infty}}(1-K)u_{K,\infty}\frac{\partial}{\partial u_{K,\infty}}\\ &\quad + \sum_{\alpha\geqslant 1}\left(Q_\alpha\frac{\partial}{\partial Q_\alpha} + \sum_{K=0}^{d_{1,\alpha}}(K+1)u_{K,\alpha}\frac{\partial}{\partial u_{K,\alpha}}\right) + \sum_j X_j\frac{\partial}{\partial X_j} + t\frac{\partial}{\partial t} + \sum_{j=1}^{g}\epsilon_j\frac{\partial}{\partial \epsilon_j}.\end{aligned} \tag{3.51}$$

Note that these formulae give other representations of the free energy in terms of its first derivatives defined independently in (3.1). Moreover, any convex linear combination will give another representation.

Proof. The formulae can be obtained by explicitly computing the residues of $\mathbf{Y}^2\mathbf{X}\,\mathrm{d}\mathbf{X}$ at the various points or by the following straightforward argument. Consider the new functions $\widetilde{\mathbf{X}} := \mathbf{X}$ and $\widetilde{\mathbf{Y}} = \mathrm{e}^c\mathbf{Y}$: the new free energy $\widetilde{\mathcal{F}}$ will be given by the same formula (3.25) in terms of the new objects. Taking $\frac{\mathrm{d}}{\mathrm{d}c}\big|_{c=0}$ gives the first formula. Some particular care has to be paid to the regularizations which involve subtraction of logarithms.

The second formula is obtained in a symmetric way. □

If we denote by $\int_\partial$ the integral operator associated with a derivative ∂, the formulae for the second-order derivatives are written concisely as

$$\partial_1\partial_2\mathcal{F} = \int_{\partial_1}\int_{\partial_2}\Omega + \delta_{\partial_1,t}\delta_{\partial_1,\partial_2}. \tag{3.52}$$

In other words[8] the Bergman kernel is the universal kernel for computing the second derivatives of the free energy and hence the two-point correlation functions of the matrix model in the planar limit.

The third-order correlation functions were computed in [2] for the case of polynomial potentials: since the reasoning is identical we only report the result. The key ingredient there

[8] We could dispose of the last term (enters only in $\partial_t^2\mathcal{F}$) by subtracting $\frac{1}{2}t^2$; this would change the t-derivative $\mu \to \mu + t$ making the formula for the first derivatives slightly different. Note that this does not affect the derivatives of order 3 and higher.

is the formula that allows you to find the variation of the Bergman kernel under infinitesimal change of the deformation parameters. The formulae can be summarized as follows:

$$(\partial\Omega)_{\mathbf{X}}(\xi,\eta) = -\int_{\rho,\partial}\sum_k \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\Omega(\xi,\zeta)\Omega(\rho,\zeta)\Omega(\eta,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} = -\sum_k \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\Omega(\xi,\zeta)\omega_\partial(\zeta)\Omega(\eta,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)}$$
$$(\partial\Omega)_{\mathbf{Y}}(\xi,\eta) = \int_{\rho,\partial}\sum_k \mathop{\mathrm{res}}_{\zeta=y_k} \frac{\Omega(\xi,\zeta)\Omega(\rho,\zeta)\Omega(\eta,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} = \sum_k \mathop{\mathrm{res}}_{\zeta=y_k} \frac{\Omega(\xi,\zeta)\omega_\partial(\zeta)\Omega(\eta,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)}, \tag{3.53}$$

where x_k and y_k denote, respectively, *all* the critical points of $\mathbf{X}$ and $\mathbf{Y}$ other than $\infty_{\mathbf{Y}}, \infty_{\mathbf{X}}$ (namely $\mathrm{d}\mathbf{X}(x_k) = 0$, $\mathrm{d}\mathbf{Y}(y_j) = 0$). These formulae follow from the Rauch variational formula [29, 22]. Note that $\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}$ in the denominator has simple poles at the ξ_j, η_j; hence the residues at these points do not contribute to the sum except for the cases where ω_∂ has a (double) pole at one of those points, namely only for the cases $\partial = \partial_{X_j}, \partial_{Y_j}$.

The final formulae for the third derivatives are simpler if we introduce the two kernels

$$\Omega^{(3)}_{\mathbf{X}}(\zeta_1,\zeta_2,\zeta_3) := -\sum_k \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\Omega(\zeta_1,\zeta)\Omega(\zeta_2,\zeta)\Omega(\zeta_3,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \tag{3.54}$$

$$\Omega^{(3)}_{\mathbf{Y}}(\zeta_1,\zeta_2,\zeta_3) := \sum_k \mathop{\mathrm{res}}_{\zeta=y_k} \frac{\Omega(\zeta_1,\zeta)\Omega(\zeta_2,\zeta)\Omega(\zeta_3,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)}. \tag{3.55}$$

This way one obtains

$$\partial_{u_{K,\alpha}}\partial_{u_{J,\beta}}\partial\mathcal{F} = \int_\partial \mathcal{U}_{K,\alpha}\mathcal{U}_{J,\beta}\Omega^{(3)}_{\mathbf{X}}, \qquad \partial_{v_{K,\alpha}}\partial_{v_{J,\beta}}\partial\mathcal{F} = \int_\partial \mathcal{V}_{K,\alpha}\mathcal{V}_{J,\beta}\Omega^{(3)}_{\mathbf{Y}} \tag{3.56}$$

$$\partial_{u_{K,\alpha}}\partial_t\partial_t\mathcal{F} = \mathcal{U}_{K,\alpha}\mathcal{T}\mathcal{T}\Omega^{(3)}_{\mathbf{X}}, \qquad \partial_{v_{J,\alpha}}\partial_t\partial_t\mathcal{F} = \mathcal{V}_{J,\alpha}\mathcal{T}\mathcal{T}\Omega^{(3)}_{\mathbf{Y}}. \tag{3.57}$$

For all other third-order derivatives, one can use either kernels

$$\partial_1\partial_2\partial_3\mathcal{F} = \int_{\partial_1}\int_{\partial_2}\int_{\partial_3}\Omega^{(3)}_{\mathbf{Y}} = \int_{\partial_1}\int_{\partial_2}\int_{\partial_3}\Omega^{(3)}_{\mathbf{X}}. \tag{3.58}$$

It should be clear to the reader that these formulae translate to *residue formulae* in the spirit of [26, 13]. For example,

$$\partial_{\epsilon_j}\partial_{\epsilon_k}\partial_{\epsilon_\ell}\mathcal{F} = -\sum_k \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\omega_j\omega_k\omega_\ell}{\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}} = \sum_k \mathop{\mathrm{res}}_{\zeta=y_k} \frac{\omega_j\omega_k\omega_\ell}{\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}}. \tag{3.59}$$

Remark 3.2. There are some superficial similarities between the free energy and the tau function of the Whitham hierarchy defined in [26]: the moduli space over which the free energy is defined indeed can be embedded as a submanifold of the moduli space considered in [26]. Nonetheless, the coordinates that are relevant to the matrix model applications are of a different nature as those introduced by Krichever. Resultingly the free energy it is not the same function: this distinction is particularly relevant in the computations of observables (i.e., derivatives) of higher order.

4. Residue formulae for higher derivatives: extended Rauch variational formulae

It is clear from the previous review of the material that in order to compute any further variation we must be able to find the variation of the kernels $\Omega^{(3)}_{\mathbf{Y}}$ and $\Omega^{(3)}_{\mathbf{X}}$: this step will produce *three* kernels

$$\Omega^{(4)}_{\mathbf{YY}}, \quad \Omega^{(4)}_{\mathbf{YX}}, \quad \Omega^{(4)}_{\mathbf{XX}}, \tag{4.1}$$

according to which variable $\mathbf{Y}$ or $\mathbf{X}$ we keep fixed under the new variation. The reason of this plethora is essentially that the variations of the basic differentials $\int_\partial \Omega$ are performed more easily either at $\mathbf{Y}$ or $\mathbf{X}$ fixed: for instance, if we compute the variation of $\omega_K = \mathcal{U}_K(\Omega)$ at $\mathbf{X}$ fixed, we obtain

$$(\partial\omega_{K,\infty})_{\mathbf{X}}(\xi) = \operatorname*{res}_{\infty_{\mathbf{X}}} \frac{\mathbf{X}^K}{K}(\partial\Omega)_{\mathbf{X}} = -\sum_k \operatorname*{res}_{\zeta=x_k} \frac{\omega_K(\zeta)\omega_\partial(\zeta)\Omega(\xi,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \tag{4.2}$$

whereas

$$(\partial\omega_{K,\infty})_{\mathbf{Y}} = \operatorname*{res}_{\infty_{\mathbf{X}}} \left(\mathbf{X}^{K-1}(\partial\mathbf{X})_{\mathbf{Y}}\Omega + \frac{\mathbf{X}^K}{K}(\partial\Omega)_{\mathbf{Y}}\right). \tag{4.3}$$

This is in fact a manifestation of the thermodynamic identity for differentials.

Lemma 4.1. *The variation of a differential at* $\mathbf{Y}$ *and* $\mathbf{X}$ *fixed are related by the following formula:*

$$(\partial\omega)_{\mathbf{Y}} = (\partial\omega)_{\mathbf{X}} + \mathrm{d}\left(\frac{\omega}{\mathrm{d}\mathbf{X}}(\partial\mathbf{X})_{\mathbf{Y}}\right) = (\partial\omega)_{\mathbf{X}} - \mathrm{d}\left(\frac{\omega\omega_\partial}{\mathrm{d}\mathbf{X}\,\mathrm{d}\mathbf{Y}}\right). \tag{4.4}$$

Proof. Writing $\omega = f\,\mathrm{d}\mathbf{Y} = g\,\mathrm{d}\mathbf{X}$, we have

$$(\partial\omega)_{\mathbf{Y}} = \left[(\partial g)_{\mathbf{X}} + \frac{\mathrm{d}g}{\mathrm{d}\mathbf{X}}(\partial\mathbf{X})_{\mathbf{Y}}\right]\mathrm{d}\mathbf{X} + g\mathrm{d}(\partial\mathbf{X})_{\mathbf{Y}} = (\partial\omega)_{\mathbf{X}} + \mathrm{d}(g(\partial\mathbf{X})_{\mathbf{Y}}). \tag{4.5}$$

Since $g = \omega/\mathrm{d}\mathbf{X}$ and $(\partial\mathbf{X})_{\mathbf{Y}} = -\omega_\partial/\mathrm{d}\mathbf{Y}$, we have the assertion. □

Using lemma 4.1 and trading the residues at the x_k over to the others (at the $y_\ell, \xi, \infty_{\mathbf{X}}$) one can check directly that formulae (4.2), (4.3) are consistent.

It should also be clear that the variation of the numerators of $\Omega^{(3)}_{\mathbf{Y},\mathbf{X}}$ are obtained by simply applying the product rule and the previously listed appropriate Rauch formulae. The only new ingredient is the variation of the denominator of $\Omega^{(3)}_{\mathbf{Y},\mathbf{X}}$ as explained below.

Suppose we want to perform a variation ∂ at $\mathbf{X}$ fixed of one of the two kernels; when we need to compute the variation of the denominator, we need a formula for $\partial\frac{1}{\mathrm{d}\mathbf{Y}}$. We should think of the expression $\frac{1}{\mathrm{d}\mathbf{Y}}$ as a meromorphic vector field on the Riemann surface and the variation is the vector field

$$\partial\left(\frac{1}{\mathrm{d}\mathbf{Y}}\right)_{\mathbf{X}} = -\frac{\mathrm{d}((\partial\mathbf{Y})_{\mathbf{X}})}{\mathrm{d}\mathbf{Y}^2} = -\frac{1}{\mathrm{d}\mathbf{Y}^2}\,\mathrm{d}\left(\frac{\omega_\partial}{\mathrm{d}\mathbf{X}}\right). \tag{4.6}$$

Now, the differential of the function $\frac{\omega_\partial}{\mathrm{d}\mathbf{X}}$ can be expressed as a residue using, once more, the Bergman kernel according to the following:

Lemma 4.2. *Let F be a (local) meromorphic function: then the differential* $\mathrm{d}F$ *can be obtained by*

$$\mathrm{d}F(\xi) = \operatorname*{res}_{\zeta=\xi} \Omega(\zeta,\xi)F(\zeta). \tag{4.7}$$

The proof is very simple using a local parameter near the point ξ and the asymptotic expansion of the Bergman kernel.

Combining lemma 4.2 with (4.6), we have the new variational formula

$$\begin{aligned} \partial\left(\frac{1}{\mathrm{d}\mathbf{Y}}\right)_{\mathbf{X}}\Bigg|_\xi &= -\frac{1}{\mathrm{d}\mathbf{Y}^2(\xi)}\operatorname*{res}_{\zeta=\xi}\Omega(\zeta,\xi)\frac{\omega_\partial(\zeta)}{\mathrm{d}\mathbf{X}(\zeta)} \\ \partial\left(\frac{1}{\mathrm{d}\mathbf{X}}\right)_{\mathbf{Y}}\Bigg|_\xi &= \frac{1}{\mathrm{d}\mathbf{X}^2(\xi)}\operatorname*{res}_{\zeta=\xi}\Omega(\zeta,\xi)\frac{\omega_\partial(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)}, \end{aligned} \tag{4.8}$$

where the different sign in the second formula is due to the fact that $(\partial\mathbf{X})_{\mathbf{Y}} = -\omega_\partial/\mathrm{d}\mathbf{Y}$.

Let us summarize the *rules of the calculus*.

(1) The variations of any differential can be performed at $\mathbf{X}$ or $\mathbf{Y}$ fixed, the two being related by lemma 4.1.

(2) The variations at $\mathbf{X}$-fixed of the vector fields $1/\mathrm{d}\mathbf{Y}$ and *vice versa* are given by equation (4.8).

(3) The variations of the Bergman bidifferential Ω are given by equations (3.53).

The choice of variable to be kept fixed $\mathbf{Y}$ versus $\mathbf{X}$ is ultimately immaterial. However, formulae can take on a significantly more involved form if one chooses the 'wrong' way of differentiation. We are going to practice this calculus and compute the fourth-order derivatives explicitly. This will also provide us with relevant formulae for the four-point correlators of the planar limit of the two-matrix model.

4.1. Fourth order

To illustrate the method, we compute the fourth derivatives with respect to $u_{K,\alpha}, u_{L,\beta}, u_{M,\gamma}, u_{N,\delta}$ (which we will denote in shorthand by the subscripts M, N, L, K only). We start from the expression for the third derivative

$$\partial_M \partial_N \partial_L \mathcal{F} := \mathcal{F}_{M,N,L} = -\sum \operatorname*{res}_{\xi=x_k} \frac{\omega_M \omega_L \omega_N}{\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}}. \tag{4.9}$$

It is quite obvious from the considerations around equation (4.2) that the extra derivative is most easily computed at $\mathbf{X}$ fixed:

$$\begin{aligned}\partial_K \mathcal{F}_{M,N,L} = &-\sum_k \operatorname*{res}_{\xi=x_k} \frac{(\partial_K \omega_M)_{\mathbf{X}} \omega_L \omega_N}{\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}} - (M \leftrightarrow L) - (M \leftrightarrow N)\\ &+ \sum \operatorname*{res}_{\xi=x_k} \frac{\omega_L \omega_M \omega_N}{\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}} \frac{\mathrm{d}(\partial_K \mathbf{Y})_{\mathbf{X}}}{\mathrm{d}\mathbf{Y}}.\end{aligned} \tag{4.10}$$

Using now lemma 4.2 and the variational formulae (3.53), we obtain

$$\partial_K \mathcal{F}_{M,N,L} = \sum_k \operatorname*{res}_{\xi=x_k} \frac{\omega_L(\xi)\omega_N(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \left(\sum_\ell \operatorname*{res}_{\zeta=x_\ell} \frac{\omega_M(\zeta)\omega_K(\zeta)\Omega(\xi,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \right)$$

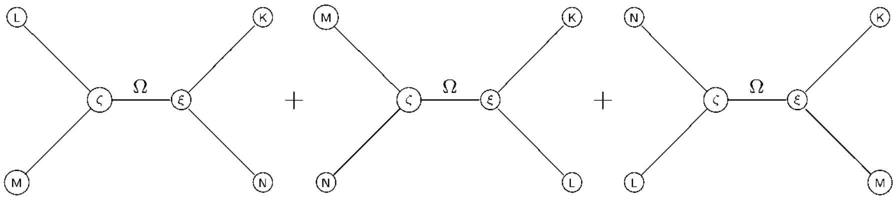
$$+ (M \leftrightarrow L) + (M \leftrightarrow N) \tag{4.11}$$

$$+ \sum \operatorname*{res}_{\xi=x_k} \frac{\omega_L(\xi)\omega_M(\xi)\omega_N(\xi)}{\mathrm{d}\mathbf{Y}(\xi)^2\,\mathrm{d}\mathbf{X}(\xi)} \operatorname*{res}_{\zeta=\xi} \frac{\omega_K(\zeta)\Omega(\zeta,\xi)}{\mathrm{d}\mathbf{X}(\zeta)}. \tag{4.12}$$

The computation could end here, since we have successfully expressed the derivatives in terms of residues of known differentials: however, this expression is not obviously symmetric in the exchange of the indices, whereas it should be since it expresses the fourth derivatives of the free energy. The expression *is symmetric*, but not at first sight. In the double sum, the order of the residues is *immaterial only for the non-diagonal part*: for the diagonal part of the sum, the residue with respect to ζ must be evaluated first. The non-diagonal part of the sum is

$$\sum_{\substack{k,\ell:\\ \ell\neq k}} \operatorname*{res}_{\xi=x_k} \operatorname*{res}_{\zeta=x_\ell} \frac{\omega_L(\xi)\omega_N(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \Omega(\xi,\zeta) \frac{\omega_M(\zeta)\omega_K(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} + (M \leftrightarrow L) + (M \leftrightarrow N), \tag{4.13}$$

where the order of the residues is, as we said, immaterial because they are taken at different points. This term corresponds diagrammatically to



and is manifestly symmetric in K, L, M, N. The diagonal part is not manifestly symmetric, but in fact we are going to show that it is. The diagonal part of the sum together with the last term is made of the following residues:

$$\mathop{\mathrm{res}}_{\xi=x_k} \frac{\omega_L(\xi)\omega_N(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\omega_M(\zeta)\omega_K(\zeta)\Omega(\xi,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} + (M \mapsto N \mapsto L)$$
$$+ \mathop{\mathrm{res}}_{\xi=x_k} \frac{\omega_L(\xi)\omega_M(\xi)\omega_N(\xi)}{\mathrm{d}\mathbf{Y}(\xi)^2\,\mathrm{d}\mathbf{X}(\xi)} \mathop{\mathrm{res}}_{\zeta=\xi} \frac{\omega_K(\zeta)\Omega(\zeta,\xi)}{\mathrm{d}\mathbf{X}(\zeta)} \tag{4.14}$$

where we stress that the residues with respect to ζ have to be evaluated *first*. For instance, a rather long computation in the local coordinate $z = \sqrt{\mathbf{X} - \mathbf{X}(x_k)}$ gives

$$\frac{1}{2}\frac{LMNK'' + LMN''K + LM''NK + L''MNK + LMNKS_B}{(\mathbf{Y}')^2} - \frac{1}{2}\frac{KLMN\mathbf{Y}'''}{(\mathbf{Y}')^3} \tag{4.15}$$

where the shorthand notation is as follows:

$$\omega_L = L(z)\,\mathrm{d}z, \qquad \omega_K = K(z)\,\mathrm{d}z, \qquad \omega_M = M(z)\,\mathrm{d}z, \qquad \omega_N = N(z)\,\mathrm{d}z \tag{4.16}$$

$$\Omega(z,z') = \left(\frac{1}{(z-z')^2} + \frac{1}{6}S_B(z,z')\right)\mathrm{d}z\,\mathrm{d}z', \tag{4.17}$$

and $S_B(z,z)$ is the *projective connection*, and all quantities are evaluated at $z = 0$.

4.1.1. Four-point correlator. This is the formal expression for

$$R^{(4)}_{4,0}(q_1,q_2,q_3,q_4) := \frac{\delta^4\mathcal{F}}{\delta V_1(q_1)V_1(q_2)V_1(q_3)V_1(q_4)}, \tag{4.18}$$

where the formal operator $\delta/\delta V_1(q)$ is defined by

$$\frac{\delta}{\delta V_1(q)} = \sum_{K=1}^{\infty} q^{-K-1}K\frac{\partial}{\partial u_{K,\infty}}. \tag{4.19}$$

By summing the four indices of the above derivatives (at least formally), we obtain

$$R^{(4)}_{4,0}(q_1,q_2,q_3,q_4)\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3\,\mathrm{d}q_4 = \Omega^{(4)}_{\mathbf{XX}}(\zeta(q_1),\zeta(q_2),\zeta(q_3),\zeta(q_4)), \tag{4.20}$$

where $\zeta(q)$ is the solution of $\mathbf{X}(\zeta) = q$ on the physical sheet of the cover $\mathbf{X} : \Sigma_g \to \mathbb{C}P^1$ and

$$\Omega^{(4)}_{\mathbf{XX}}(1,2,3,4) = \sum_r \mathop{\mathrm{res}}_{\xi=x_k} \frac{\Omega(1,\xi)\Omega(2,\xi)\Omega(3,\xi)}{\mathrm{d}\mathbf{Y}^2(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \mathop{\mathrm{res}}_{\zeta=\xi} \Omega(\zeta,\xi)\frac{\Omega(\zeta,4)}{\mathrm{d}\mathbf{X}(\zeta)} \tag{4.21}$$

$$+ \sum_r \mathop{\mathrm{res}}_{\xi=x_k} \sum_k \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\Omega(1,\zeta)\Omega(4,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)}\Omega(\zeta,\xi)\frac{\Omega(2,\xi)\Omega(3,\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3). \tag{4.22}$$

Note that this kernel is symmetric in the four variables although not at first sight, but by the same considerations as before. In a similar way one can obtain the other four-point correlator

$$R^{(4)}_{0,4}(p_1,p_2,p_3,p_4) := \frac{\delta^4\mathcal{F}}{\delta V_2(p_1)V_2(p_2)V_2(p_3)V_2(p_4)}, \tag{4.23}$$

where $\delta/\delta V_2(p)$ is defined similarly as before by

$$\frac{\delta}{\delta V_2(p)} := \sum_{J=1}^{\infty} J p^{-J-1} \frac{\partial}{\partial v_J}. \tag{4.24}$$

The derivation of the formula is completely parallel; hence we only give the final result

$$R^{(4)}_{0,4}(p_1, p_2, p_3, p_4)\, \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p_3\, \mathrm{d}p_4 = \Omega^{(4)}_{\mathbf{YY}}(\xi(p_1), \xi(p_2), \xi(p_3), \xi(p_4)), \tag{4.25}$$

where $\xi(p)$ is the solution of $\mathbf{Y}(\xi) = p$ on the physical sheet of the cover $\mathbf{Y} : \Sigma_g \to \mathbb{C}P^1$ and

$$\Omega^{(4)}_{\mathbf{YY}}(1, 2, 3, 4) = \sum_{\ell} \operatorname*{res}_{\xi=y_\ell} \frac{\Omega(1, \xi)\Omega(2, \xi)\Omega(3, \xi)}{\mathrm{d}\mathbf{Y}(\xi)\, \mathrm{d}\mathbf{X}^2(\xi)} \operatorname*{res}_{\zeta=\xi} \Omega(\zeta, \xi) \frac{\Omega(\zeta, 4)}{\mathrm{d}\mathbf{Y}(\zeta)} \tag{4.26}$$

$$+ \sum_{\ell} \operatorname*{res}_{\xi=y_\ell} \sum_{s} \operatorname*{res}_{\zeta=y_s} \frac{\Omega(1, \zeta)\Omega(4, \zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\, \mathrm{d}\mathbf{X}(\zeta)} \Omega(\zeta, \xi) \frac{\Omega(2, \xi)\Omega(3, \xi)}{\mathrm{d}\mathbf{Y}(\xi)\, \mathrm{d}\mathbf{X}(\xi)} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3). \tag{4.27}$$

4.2. 'Mixed' fourth-order derivatives

As a further example, we compute the derivatives with respect to u_L, u_M, v_N, v_K: we leave the derivative with respect to v_K last and perform it at $\mathbf{X}$ fixed

$$\mathcal{F}_{LM\tilde{N}\tilde{K}} = \partial_{\tilde{K}} \mathcal{F}_{LM\tilde{N}} = \partial_{\tilde{K}} \sum \operatorname*{res}_{x_k} \frac{\omega_L \omega_M \omega_{\widetilde{N}}}{\mathrm{d}\mathbf{Y}\, \mathrm{d}\mathbf{X}} \tag{4.28}$$

$$= \sum \operatorname*{res}_{x_k} \left(\frac{(\partial_{\widetilde{K}}\omega_L)_{\mathbf{X}} \omega_M \omega_{\widetilde{N}} + \omega_L (\partial_{\widetilde{K}}\omega_M)_{\mathbf{X}} \omega_{\widetilde{N}} + \omega_M \omega_L (\partial_{\widetilde{K}}\omega_{\widetilde{N}})_{\mathbf{Y}} + \omega_M \omega_L \mathrm{d}\left(\frac{\omega_{\widetilde{N}}\omega_{\widetilde{K}}}{\mathrm{d}\mathbf{Y}\,\mathrm{d}\mathbf{X}}\right)}{\mathrm{d}\mathbf{Y}\, \mathrm{d}\mathbf{X}} \right.$$

$$\left. - \frac{\omega_L \omega_M \omega_{\widetilde{N}}}{\mathrm{d}\mathbf{Y}\, \mathrm{d}\mathbf{X}} \frac{\mathrm{d}(\partial_{\widetilde{K}}\mathbf{Y})_{\mathbf{X}}}{\mathrm{d}\mathbf{Y}} \right) \tag{4.29}$$

$$= \sum \operatorname*{res}_{x_k} \left[\frac{(\partial_{\widetilde{K}}\omega_L)_{\mathbf{X}} \omega_M \omega_{\widetilde{N}} + \omega_L (\partial_{\widetilde{K}}\omega_M)_{\mathbf{X}} \omega_{\widetilde{N}} + \omega_M \omega_L (\partial_{\widetilde{K}}\omega_{\widetilde{N}})_{\mathbf{Y}}}{\mathrm{d}\mathbf{Y}\, \mathrm{d}\mathbf{X}} \right.$$

$$\left. + \frac{\omega_M \omega_L}{\mathrm{d}\mathbf{Y}\, \mathrm{d}\mathbf{X}} \mathrm{d}\left(\frac{\omega_{\widetilde{N}}\omega_{\widetilde{K}}}{\mathrm{d}\mathbf{Y}\, \mathrm{d}\mathbf{X}}\right) - \frac{\omega_L \omega_M \omega_{\widetilde{N}}}{\mathrm{d}\mathbf{Y}^2\, \mathrm{d}\mathbf{X}} \mathrm{d}\left(\frac{\omega_{\widetilde{K}}}{\mathrm{d}\mathbf{X}}\right) \right] \tag{4.30}$$

$$= \sum \operatorname*{res}_{\zeta=x_k} \frac{\omega_M(\zeta)\omega_L(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\, \mathrm{d}\mathbf{X}(\zeta)} \sum \operatorname*{res}_{\xi=y_\ell} \Omega(\xi, \zeta) \frac{\omega_{\widetilde{N}}(\xi)\omega_{\widetilde{K}}(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\, \mathrm{d}\mathbf{X}(\xi)} \tag{4.31}$$

$$- \sum \operatorname*{res}_{\zeta=x_k} \left(\frac{\omega_M(\zeta)\omega_{\widetilde{N}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\, \mathrm{d}\mathbf{X}(\zeta)} \sum \operatorname*{res}_{\xi=x_\ell} \frac{\omega_L(\xi)\omega_{\widetilde{K}}(\xi)\Omega(\xi, \zeta)}{\mathrm{d}\mathbf{Y}(\xi)\, \mathrm{d}\mathbf{X}(\xi)} + (L \leftrightarrow M) \right) \tag{4.32}$$

$$+ \sum \operatorname*{res}_{\zeta=x_k} \left[\frac{\omega_M(\zeta)\omega_L(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\, \mathrm{d}\mathbf{X}(\zeta)} \mathrm{d}\left(\frac{\omega_{\widetilde{N}}(\zeta)\omega_{\widetilde{K}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\, \mathrm{d}\mathbf{X}(\zeta)}\right) - \frac{\omega_L(\zeta)\omega_M(\zeta)\omega_{\widetilde{N}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)^2\, \mathrm{d}\mathbf{X}(\zeta)} \mathrm{d}\left(\frac{\omega_{\widetilde{K}}(\zeta)}{\mathrm{d}\mathbf{X}(\zeta)}\right) \right]. \tag{4.33}$$

Note that in order to compute effectively the derivative of $\omega_{\widetilde{K}}$ at $\mathbf{X}$-fixed, we have used lemma 4.1. Once more one can check that the resulting expression is symmetric in $\widetilde{N} \leftrightarrow \widetilde{K}$ and $M \leftrightarrow L$. The only terms which do not have this symmetry at first sight are the diagonal part of the double sum over x_k together with the last term:

$$\mathcal{F}_{LM\tilde{N}\tilde{K}} = \sum \operatorname*{res}_{\zeta=x_k} \sum \operatorname*{res}_{\xi=y_\ell} \frac{\omega_M(\zeta)\omega_L(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\, \mathrm{d}\mathbf{X}(\zeta)} \Omega(\xi, \zeta) \frac{\omega_{\widetilde{N}}(\xi)\omega_{\widetilde{K}}(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\, \mathrm{d}\mathbf{X}(\xi)} \tag{4.34}$$

$$-\sum_{\substack{k,\ell:\\ \ell\neq k}} \mathop{\mathrm{res}}_{\zeta=x_\ell} \mathop{\mathrm{res}}_{\xi=x_k} \left(\frac{\omega_M(\zeta)\omega_{\widetilde{N}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \Omega(\xi,\zeta) \frac{\omega_L(\xi)\omega_{\widetilde{K}}(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} + (L \leftrightarrow M) \right) \tag{4.35}$$

$$+\sum \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\omega_M(\zeta)\omega_L(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \mathrm{d}\left(\frac{\omega_{\widetilde{N}}(\zeta)\omega_{\widetilde{K}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \right) \tag{4.36}$$

$$-\sum_k \mathop{\mathrm{res}}_{\zeta=x_k} \left(\mathop{\mathrm{res}}_{\xi=x_k} \frac{\omega_M(\zeta)\omega_{\widetilde{N}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \Omega(\xi,\zeta) \frac{\omega_L(\xi)\omega_{\widetilde{K}}(\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} + \frac{\omega_L(\zeta)\omega_M(\zeta)\omega_{\widetilde{N}}(\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)^2\,\mathrm{d}\mathbf{X}(\zeta)} \mathrm{d}\left(\frac{\omega_{\widetilde{K}}(\zeta)}{\mathrm{d}\mathbf{X}(\zeta)} \right) \right). \tag{4.37}$$

The same considerations about symmetry done previously apply to the sum on line (4.37) as well.

4.2.1. Four-point correlator. Using the above computation, we can compute the following four-point correlator:

$$R^{(4)}_{2,2}(q_1,q_2,p_1,p_2) := \frac{\delta^4\mathcal{F}}{\delta V_1(q_1)V_1(q_2)V_2(p_1)V_2(p_2)}. \tag{4.38}$$

Performing the multiple summation, we find

$$R^{(4)}_{2,2}(q_1,q_2,p_1,p_2)\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}p_1\,\mathrm{d}p_2 = \Omega^{(4)}_{\mathbf{XY}}(\zeta(q_1),\zeta(q_2),\xi(p_1),\xi(p_2)), \tag{4.39}$$

where $\xi(p)$ is the solution on the physical sheet of $\mathbf{Y}(\xi) = p$ and

$$\Omega^{(4)}_{\mathbf{XY}}(1,2,\widetilde{1},\widetilde{2}) := \sum \mathop{\mathrm{res}}_{\zeta=x_k} \sum \mathop{\mathrm{res}}_{\xi=y_\ell} \frac{\Omega(1,\zeta)\Omega(2,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \Omega(\zeta,\xi) \frac{\Omega(\widetilde{1},\xi)\Omega(\widetilde{2},\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \tag{4.40}$$

$$-\sum_{\substack{k,r\\ k\neq r}} \mathop{\mathrm{res}}_{\zeta=x_r} \mathop{\mathrm{res}}_{\xi=x_k} \frac{\Omega(1,\zeta)\Omega(\widetilde{1},\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \Omega(\zeta,\xi) \frac{\Omega(2,\xi)\Omega(\widetilde{2},\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} - (1 \leftrightarrow 2) \tag{4.41}$$

$$+\sum \mathop{\mathrm{res}}_{\zeta=x_k} \frac{\Omega(1,\zeta)\Omega(2,\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \mathrm{d}_\zeta \frac{\Omega(\widetilde{1},\zeta)\Omega(\widetilde{2},\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \tag{4.42}$$

$$-\sum \mathop{\mathrm{res}}_{\zeta=x_k} \left(\mathop{\mathrm{res}}_{\xi=x_k} \frac{\Omega(1,\zeta)\Omega(\widetilde{1},\zeta)}{\mathrm{d}\mathbf{Y}(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \Omega(\zeta,\xi) \frac{\Omega(2,\xi)\Omega(\widetilde{2},\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} + \frac{\Omega(1,\zeta)\Omega(2,\zeta)\Omega(\widetilde{1},\zeta)}{\mathrm{d}\mathbf{Y}^2(\zeta)\,\mathrm{d}\mathbf{X}(\zeta)} \mathrm{d}_\zeta \left(\frac{\Omega(\widetilde{2},\zeta)}{\mathrm{d}\mathbf{X}(\zeta)} \right) \right). \tag{4.43}$$

Repeating the derivation from the beginning, one can realize that there is no need of any other kernel for

$$R^{(4)}_{3,1}(q_1,q_2,q_3,p_2) := \frac{\delta^4\mathcal{F}}{\delta V_1(q_1)V_1(q_2)V_1(p_3)V_2(p_1)}, \tag{4.44}$$

which is given by

$$R^{(4)}_{3,1}(q_1,q_2,q_3,p_1)\,\mathrm{d}q_1\mathrm{d}q_2\,\mathrm{d}q_3\,\mathrm{d}p_1 = \Omega^{(4)}_{\mathbf{XX}}(\zeta(q_1),\zeta(q_2),\zeta(q_3),\xi(p_1)). \tag{4.45}$$

4.2.2. Summary of all fourth derivatives. These three kernels are sufficient for us to write all fourth derivatives compactly as some new *residue formulae* (note: the order in which the integral operators appear is to mean that they are applied to the variable that appear in the corresponding position in the kernel)

$$\partial_{u_K}\partial_{u_J}\partial_{v_L}\partial_{v_M}\mathcal{F} = \mathcal{U}_K\mathcal{U}_J\mathcal{V}_L\mathcal{V}_M\Omega^{(4)}_{\mathbf{XY}} \tag{4.46}$$

$$\partial_{u_K}\partial_{v_J}\partial_t^2\mathcal{F} = \mathcal{U}_K\mathcal{T}\mathcal{T}\mathcal{V}_J\Omega^{(4)}_{\mathbf{XY}} \tag{4.47}$$

$$\partial_{u_K}\partial_1\partial_2\partial_3\mathcal{F} = \mathcal{U}_K\int_{\partial_1}\int_{\partial_2}\int_{\partial_3}\Omega^{(4)}_{\mathbf{XX}} \tag{4.48}$$

$$\partial_{v_J}\partial_1\partial_2\partial_3\mathcal{F} = \mathcal{V}_J\int_{\partial_1}\int_{\partial_2}\int_{\partial_3}\Omega^{(4)}_{\mathbf{YY}} \tag{4.49}$$

$$\partial_1\partial_2\partial_3\partial_4\mathcal{F} = \int_{\partial_1}\int_{\partial_2}\int_{\partial_3}\int_{\partial_4}\Omega^{(4)}_{\mathbf{YY}} = \int_{\partial_1}\int_{\partial_2}\int_{\partial_3}\int_{\partial_4}\Omega^{(4)}_{\mathbf{XX}}, \tag{4.50}$$

where the symbols ∂_j here mean derivatives with respect to variables not included in the previous items of the list.

4.3. Higher-order correlators

The computation of any derivative of any order is just a matter of application of the 'rules of calculus' outlined previously; in this fashion one could obtain residue formulae for any derivative and possibly develop some diagrammatic rules to help in the computation. We leave this exercise to the reader who may need it for his/her application to a specific problem. The formal 'puncture' operators

$$\mathrm{d}\mathbf{X}(\xi)\frac{\delta}{\delta V_1(\mathbf{X}(\xi))}, \qquad \mathrm{d}\mathbf{Y}(\xi)\frac{\delta}{\delta V_2(\mathbf{Y}(\xi))} \tag{4.51}$$

act as follows on each term:

$$\mathrm{d}\mathbf{X}(1)\frac{\delta\Omega(2,3)}{\delta V_1(\mathbf{X}(1))} = \sum\operatorname*{res}_{\xi=x_k}\frac{\Omega(1,\xi)\Omega(2,\xi)\Omega(3,\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \tag{4.52}$$

$$\mathrm{d}\mathbf{X}(1)\frac{\delta}{\delta V_1(\mathbf{X}(1))}\left(\frac{1}{\mathrm{d}\mathbf{Y}(2)}\right) = \frac{1}{\mathrm{d}\mathbf{Y}^2(2)}\mathrm{d}_2\left(\frac{\Omega(1,2)}{\mathrm{d}\mathbf{X}(2)}\right) = \frac{1}{\mathrm{d}\mathbf{Y}^2(2)}\operatorname*{res}_{\xi=2}\frac{\Omega(2,\xi)\Omega(\xi,1)}{\mathrm{d}\mathbf{X}(\xi)} \tag{4.53}$$

$$\mathrm{d}\mathbf{Y}(1)\frac{\delta\Omega(2,3)}{\delta V_2(\mathbf{Y}(1))} = -\sum\operatorname*{res}_{\xi=y_k}\frac{\Omega(1,\xi)\Omega(2,\xi)\Omega(3,\xi)}{\mathrm{d}\mathbf{Y}(\xi)\,\mathrm{d}\mathbf{X}(\xi)} \tag{4.54}$$

$$\mathrm{d}\mathbf{Y}(1)\frac{\delta}{\delta V_2(\mathbf{Y}(1))}\left(\frac{1}{\mathrm{d}\mathbf{X}(2)}\right) = -\frac{1}{\mathrm{d}\mathbf{Y}^2(2)}\mathrm{d}_2\left(\frac{\Omega(1,2)}{\mathrm{d}\mathbf{Y}(2)}\right) = -\frac{1}{\mathrm{d}\mathbf{X}^2(2)}\operatorname*{res}_{\xi=2}\frac{\Omega(2,\xi)\Omega(\xi,1)}{\mathrm{d}\mathbf{Y}(\xi)} \tag{4.55}$$

Combining these 'rules' it is easy to obtain any correlator: the resulting expression *will* be symmetric in the exchange of the variables, although to recognize this some careful analysis of the residues is required.

4.4. The equilibrium correlators

The derivation of the multiple derivatives of the equilibrium free energy $\mathcal{G}$ follows the same lines and the results are the same formulae with Ω replaced by $\widetilde{\Omega}$ (clearly there are no derivatives with respect to the filling fractions ϵ_j which are now dependent functions). In general, the rules of calculus for $\mathcal{G}$ are the same as the rule of calculus for $\mathcal{F}$ with all the instances of the Bergman kernel replaced by the dual kernel $\widetilde{\Omega}$.

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Appendix A. Explicit form of the regularized integrals

In this section we provide explicit formulae for the regularized integrals used in the definition of the free energy and the τ function of the previous section.

The main tools are the following properties which were used in the proof of the derivatives of the free energy:

$$\begin{aligned}\mathbf{Y}\,\mathrm{d}\mathbf{X} = \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}(\Omega) + \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}(\Omega)\\ + t\mathcal{T}(\Omega) + \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j(\Omega) - \sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{XY}\Omega\end{aligned} \tag{A.1}$$

$$\begin{aligned}\mathbf{X}\,\mathrm{d}\mathbf{Y} = -\sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}(\Omega) - \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}(\Omega)\\ - t\mathcal{T}(\Omega) - \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j(\Omega) - \sum_{\zeta\in\mathcal{D}_{\mathbf{X}}}\operatorname*{res}_{\zeta}\mathbf{XY}\Omega.\end{aligned} \tag{A.2}$$

Let us compute $⨍_{q_\alpha}^{\infty_{\mathbf{X}}}\mathbf{Y}\,\mathrm{d}\mathbf{X}$ according to the original definition of regularization: since the operator $⨍_{q_\alpha}^{\infty_{\mathbf{X}}}$ commutes with the integral operators/regularizations in (A.1), we obtain immediately

$$\begin{aligned}⨍_{q_\alpha}^{\infty_{\mathbf{X}}}\mathbf{Y}\,\mathrm{d}\mathbf{X} = \sum_{\alpha=0}\sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha}\mathcal{U}_{K,\alpha}\left(\int_{q_\alpha}^{\infty_{\mathbf{X}}}\Omega\right) + \sum_{\alpha=0}\sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha}\mathcal{V}_{J,\alpha}\left(\int_{q_\alpha}^{\infty_{\mathbf{X}}}\Omega\right)\\ + t\mathcal{T}\left(\int_{q_\alpha}^{\infty_{\mathbf{X}}}\Omega\right) + \sum_{j=1}^{g}\epsilon_j\mathcal{E}_j\left(\int_{q_\alpha}^{\infty_{\mathbf{X}}}\Omega\right) - \sum_{\zeta\in\mathcal{D}_{\mathbf{Y}}}\operatorname*{res}_{\zeta}\mathbf{XY}\int_{q_\alpha}^{\infty_{\mathbf{X}}}\Omega.\end{aligned} \tag{A.3}$$

The differential $\int_{\infty_{\mathbf{X}}}^{q_\alpha}\Omega$ is the unique normalized differential of the third kind with simple poles at q_α, $\infty_{\mathbf{X}}$ and residues, respectively, +1, −1. To simplify formulae let us define for any two points ξ, η the following function:

$$\Lambda_{\xi,\eta}(\zeta) := \exp\left[\int_{\zeta_0}^{\zeta}\int_{\xi}^{\eta}\Omega\right], \qquad \int_{\xi}^{\eta}\Omega = \frac{\mathrm{d}\Lambda_{\xi,\eta}}{\Lambda_{\xi,\eta}}. \tag{A.4}$$

This is a multivalued function around the b-cycles; on the simply-connected domain obtained by dissection of our surface, $\Lambda_{\xi,\eta}$ has a simple pole at ξ and a simple zero at η. It is defined up to a multiplicative constant (depending on the base point for the outer integration), which however will not affect our result. With this definition we have ($q_0 := \infty_{\mathbf{X}}$, $p_0 := \infty_{\mathbf{Y}}$)

$$\partial_{u_{0,\alpha}}\mathcal{F} = -\!\!\!\!\!\!\int_{q_\alpha}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} = -\sum_{\widetilde{\alpha}=0} \operatorname*{res}_{q_{\widetilde{\alpha}}} V_{1,\widetilde{\alpha}}(\mathbf{X})\frac{\mathrm{d}\Lambda}{\Lambda} + \sum_{\beta=0}\operatorname*{res}_{q_\beta}(V_{2,\beta}(\mathbf{Y}) - \mathbf{XY})\frac{\mathrm{d}\Lambda}{\Lambda}$$
$$+ \sum_{\widetilde{\alpha}\neq\{\alpha,0\}} u_{0,\widetilde{\alpha}} \ln\left(\frac{\gamma_{\infty_{\mathbf{X}}}}{\Lambda(q_{\widetilde{\alpha}})}\right) + u_{0,\alpha}\ln\left(\frac{\gamma_{\infty_{\mathbf{X}}}}{\gamma_{q_\alpha}}\right)$$
$$+ \sum_{\beta=1} v_{0,\beta}\ln\left(\frac{\Lambda(p_\beta)}{\Lambda(\infty_{\mathbf{Y}})}\right) + t\ln\left(\frac{\gamma_{\infty_{\mathbf{X}}}}{\Lambda(\infty_{\mathbf{Y}})}\right) + \sum_{j=1}^{g}\frac{\epsilon_j}{2\mathrm{i}\pi}\oint_{b_j}\frac{\mathrm{d}\Lambda}{\Lambda} \tag{A.5}$$

where we have set $\Lambda := \Lambda_{q_\alpha,\infty_{\mathbf{X}}}$ and

$$\ln\gamma_{\infty_{\mathbf{X}}} := \lim_{\epsilon\to\infty_{\mathbf{X}}} \ln\left(\Lambda_{q_\alpha,\infty_{\mathbf{X}}}\mathbf{X}\right) \qquad \ln\gamma_{q_\alpha} := \lim_{\epsilon\to q_\alpha}\ln\left(\Lambda_{q_\alpha,\infty_{\mathbf{X}}}(\mathbf{X} - Q_\alpha)\right). \tag{A.6}$$

The formulae for the derivatives with respect to $v_{0,\alpha}$ are obtained by interchanging all the roles of $\mathbf{X}$, $\infty_{\mathbf{X}}$, q_α with $\mathbf{Y}$, $\infty_{\mathbf{Y}}$, p_α.

Finally the formula for the t-derivative

$$\partial_t\mathcal{F} = -\!\!\!\!\!\!\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}} \mathbf{Y}\,\mathrm{d}\mathbf{X} - \sum_\alpha v_{0,\alpha} = -\sum_{\widetilde{\alpha}=0}\operatorname*{res}_{q_{\widetilde{\alpha}}} V_{1,\widetilde{\alpha}}(\mathbf{X})\frac{\mathrm{d}\Lambda}{\Lambda} + \sum_{\beta=0}\operatorname*{res}_{q_\beta}(V_{2,\beta}(\mathbf{Y}) - \mathbf{XY})\frac{\mathrm{d}\Lambda}{\Lambda}$$
$$+ \sum_{\widetilde{\alpha}=1} u_{0,\widetilde{\alpha}}\ln\left(\frac{\gamma_{\infty_{\mathbf{X}}}}{\Lambda(q_{\widetilde{\alpha}})}\right) + \sum_{\beta=1} v_{0,\beta}\ln\left(\frac{\Lambda(p_\beta)}{\gamma_{\infty_{\mathbf{Y}}}}\right) + t\ln\left(\frac{\gamma_{\infty_{\mathbf{X}}}}{\gamma_{\infty_{\mathbf{Y}}}}\right) + \sum_{j=1}^{g}\frac{\epsilon_j}{2\mathrm{i}\pi}\oint_{b_j}\frac{\mathrm{d}\Lambda}{\Lambda} + t \tag{A.7}$$

where, this time,

$$\Lambda := \Lambda_{\infty_{\mathbf{Y}},\infty_{\mathbf{X}}} \qquad \ln(\gamma_{\infty_{\mathbf{X}}}) := \lim_{\epsilon\to\infty_{\mathbf{X}}}\ln(\Lambda\mathbf{X}) \qquad \ln(\gamma_{\infty_{\mathbf{Y}}}) := \lim_{\epsilon\to\infty_{\mathbf{Y}}}\ln\left(\frac{\Lambda}{\mathbf{Y}}\right). \tag{A.8}$$

The extra '$\sum_\alpha v_{0,\alpha}$' which cancels with the same term in the expression for μ is due to a careful analysis of the regularization prescription for the following term in the computation:

$$\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{XY}\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\Omega = \lim_{\epsilon\to\infty_{\mathbf{Y}}}\left(\int_{\epsilon}^{\infty_{\mathbf{X}}}\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{XY}\Omega + \mathbf{X}(\epsilon)\mathbf{Y}(\epsilon)\right) = -t - \sum_\alpha v_{0,\alpha} + -\!\!\!\!\!\!\int_{\infty_{\mathbf{Y}}}^{\infty_{\mathbf{X}}}\operatorname*{res}_{\infty_{\mathbf{Y}}}\mathbf{XY}\Omega. \tag{A.9}$$

Note that, in all these formulae, the b-periods of $\frac{\mathrm{d}\Lambda}{\Lambda}$ are the Abel map of the two poles of this differential.

Appendix B. Example: one cut case (genus zero) and conformal maps

The formulae for the derivatives simplify drastically in case the curve Σ_g is a rational curve. In this case, introducing a global coordinate λ (as explained in [1, 2]) with a zero at $\infty_{\mathbf{Y}}$ and a pole at $\infty_{\mathbf{X}}$ and suitably normalized one can always write the two functions $\mathbf{X}$, $\mathbf{Y}$ as

$$\begin{aligned}\mathbf{X} &= \gamma\lambda + \sum_{K=0}^{d_{2,\infty}} A_{K,\infty}\lambda^{-K} + \sum_\alpha\sum_{K=0}^{d_{2,\alpha}} A_{K,\alpha}(\lambda - \lambda_{\widetilde{\alpha}})^{-K-1} + \sum_{j=1}^{s}\frac{F_j}{\lambda - \lambda_{j,Y}}\\ \mathbf{Y} &= \frac{\gamma}{\lambda} + \sum_{J=0}^{d_{1,\infty}} B_{j,\infty}\lambda^{j} + \sum_\alpha\sum_{J=0}^{d_{1,\alpha}} B_{J,\alpha}(\lambda - \lambda_\alpha)^{-J-1} + \sum_{j=1}^{r}\frac{G_j}{\lambda - \lambda_{j,X}}\end{aligned} \tag{B.1}$$

$$Q_\alpha := \mathbf{X}(\lambda_\alpha), \qquad P_{\widetilde{\alpha}} := \mathbf{Y}(\lambda_{\widetilde{\alpha}}). \tag{B.2}$$

The parameters γ, A_j, F_j and $B_j, G_j, j = 1, \ldots, s$ are not independent but are constrained by the following set of linear equations (in $\gamma, A_j, B_j, F_j, G_j$):

$$\mathbf{X}'(\lambda_{j,X}) = 0,\ j = 1, \ldots, r, \qquad \mathbf{Y}'(\lambda_{j,Y}) = 0,\ j = 1, \ldots, s. \tag{B.3}$$

As we have already mentioned, the equivalent of the Bergman kernel is simply

$$\Omega(\lambda, \mu) = \frac{\mathrm{d}\lambda\,\mathrm{d}\mu}{(\lambda - \mu)^2}. \tag{B.4}$$

This is the kernel of the derivative followed by projection to the principal part. For example, the differentials $\omega_{K,\infty}$

$$\omega_{K,\infty}(\lambda) = -\mathop{\mathrm{res}}_{\mu=\infty} \frac{\mathbf{X}^K(\mu)}{K}\Omega(\lambda, \mu) = \frac{1}{K}(\mathbf{X}^K)_+\,\mathrm{d}\lambda, \tag{B.5}$$

where the $\pm$ subscripts mean the polynomial or the Laurent part of the expression enclosed in the brackets. Similar completely explicit formulae for all other differentials in the lists (3.13)–(3.15) are left to the reader.

The coordinates are given by the usual formulae (2.3). The free energy can be written in a quite explicit form using the following simplifications due to the existence of a global coordinate $\lambda(\lambda_0 := \infty, \lambda_{\widetilde{0}} := 0)$:

$$\begin{aligned}
\partial_{u_{0,\alpha}}\mathcal{F} = &-\sum_{\beta\geqslant 0}\mathop{\mathrm{res}}_{\lambda=\lambda_\beta} V_1(\mathbf{X})\frac{\mathrm{d}\lambda}{\lambda-\lambda_\alpha} + \sum_{\widetilde{\beta}\geqslant 0}\mathop{\mathrm{res}}_{\lambda=\lambda_{\widetilde{\beta}}}(V_2(\mathbf{Y}) - \mathbf{XY})\frac{\mathrm{d}\lambda}{\lambda-\lambda_\alpha}\\
&+\sum_{\beta\neq\{\alpha,0\}} u_{0,\beta}\ln(\gamma(\lambda_\beta-\lambda_\alpha)) + u_{0,\alpha}\ln\left(\frac{\gamma}{\mathbf{X}'(\lambda_\alpha)}\right)\\
&+\sum_{\widetilde{\beta}\geqslant 0} v_{0,\beta}\ln\left(\frac{\lambda_\alpha}{\lambda_\alpha-\lambda_{\widetilde{\beta}}}\right) + t\ln(\lambda_\alpha\gamma)
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\partial_{v_{0,\widetilde{\alpha}}}\mathcal{F} = &-\sum_{\widetilde{\beta}\geqslant 0}\mathop{\mathrm{res}}_{\lambda=\lambda_{\widetilde{\beta}}} V_{2,\widetilde{\beta}}(\mathbf{Y})\frac{\lambda_{\widetilde{\alpha}}\,\mathrm{d}\lambda}{\lambda(\lambda_{\widetilde{\alpha}}-\lambda)} + \sum_{\beta\geqslant 0}\mathop{\mathrm{res}}_{\lambda=\lambda_\beta}(V_1(\mathbf{X}) - \mathbf{XY})\frac{\lambda_{\widetilde{\alpha}}\,\mathrm{d}\lambda}{\lambda(\lambda_{\widetilde{\alpha}}-\lambda)}\\
&-\sum_{\widetilde{\beta}\neq\{\widetilde{\alpha},0\}} v_{0,\widetilde{\beta}}\ln\left(\frac{(\lambda_{\widetilde{\alpha}})^2}{(\lambda_{\widetilde{\beta}}-\lambda_{\widetilde{\alpha}})\gamma}\right) - v_{0,\widetilde{\alpha}}\ln\left(\frac{(\lambda_{\widetilde{\alpha}})^2\mathbf{Y}'(\lambda_{\widetilde{\alpha}})}{\gamma}\right)\\
&+\sum_{\beta\geqslant 0} u_{0,\beta}\ln\left(\frac{\lambda_\beta}{\lambda_\beta-\lambda_{\widetilde{\alpha}}}\right) + t\ln\left(\frac{\gamma}{\lambda_{\widetilde{\alpha}}}\right)
\end{aligned} \tag{B.7}$$

since $\Lambda_{q_\alpha,\infty_\mathbf{X}} = \frac{1}{\lambda-\lambda_\alpha}$ and $\Lambda_{p_{\widetilde{\alpha}},\infty_\mathbf{Y}} = \frac{\lambda}{\lambda-\lambda_{\widetilde{\alpha}}}$. Moreover, using this time $\Lambda_{\infty_\mathbf{X},\infty_\mathbf{Y}} = \lambda$ and formula (A.7)

$$\begin{aligned}
\partial_t\mathcal{F} = &\sum_{\beta\geqslant 0}\mathop{\mathrm{res}}_{\lambda=\lambda_\beta} V_{1,\beta}(\mathbf{X})\frac{\mathrm{d}\lambda}{\lambda} - \sum_{\widetilde{\beta}\geqslant 0}\mathop{\mathrm{res}}_{\lambda=\lambda_{\widetilde{\beta}}}(V_{2,\widetilde{\beta}}(\mathbf{Y}) - \mathbf{XY})\frac{\mathrm{d}\lambda}{\lambda}\\
&+\sum_{\beta=1} u_{0,\beta}\ln(\lambda_\beta\gamma) - \sum_{\widetilde{\beta}=1} v_{0,\beta}\ln\left(\frac{\lambda_\beta}{\gamma}\right) + t\ln(\gamma^2) + t.
\end{aligned} \tag{B.8}$$

By computing the other residues, one can get explicit formulae for the free energy in terms of the uniformization (B.1) and using theorem 3.5.

Denoting as before by x_k and y_ℓ the critical points of the functions $\mathbf{X}$ and $\mathbf{Y}$,[9] respectively, we have as example of fourth-point correlators

$$- R^{(4)}_{4,0}(\mu_1,\mu_2,\mu_3,\mu_4)\mathbf{X}'(\mu_1)\mathbf{X}'(\mu_2)\mathbf{X}'(\mu_2)\mathbf{X}'(\mu_4) = \sum_{\substack{k,r\\k\neq r}} \frac{(\mu_1-x_k)^{-2}(\mu_2-x_k)^{-2}}{\mathbf{Y}'(x_k)\mathbf{X}''(x_k)}$$

$$\times\frac{1}{(x_k-x_r)^2}\frac{(\mu_3-x_r)^{-2}(\mu_4-x_r)^{-2}}{\mathbf{Y}'(x_r)\mathbf{X}''(x_r)} + (\mu_1\leftrightarrow\mu_3)+(\mu_1\leftrightarrow\mu_4) \tag{B.9}$$

$$+\sum_k \frac{1}{6\mathbf{Y}'''(\mathbf{X}'')^4}\left[\frac{(2\mathbf{Y}'(\mathbf{X}''')^2+2\mathbf{Y}''\mathbf{X}''\mathbf{X}'''-3(\mathbf{X}'')^2\mathbf{Y}'''-3\mathbf{X}^{(iv)}\mathbf{X}''\mathbf{Y}')}{(\mu_1-x_k)^2(\mu_2-x_k)^2(\mu_3-x_k)^2(\mu_4-x_k)^2}\right. \tag{B.10}$$

$$+\frac{18(\mathbf{X}'')^2\mathbf{Y}'((\mu_1-x_k)^{-2}+cyc)}{(\mu_1-x_k)^2(\mu_2-x_k)^2(\mu_3-x_k)^2(\mu_4-x_k)^2}$$

$$\left.\left.-\frac{2\mathbf{X}'''\mathbf{X}''\mathbf{Y}'((\mu_1-x_k)^{-1}+cyc)}{(\mu_1-x_k)^2(\mu_2-x_k)^2(\mu_3-x_k)^2(\mu_4-x_k)^2}\right]\right|_{\lambda=x_k}. \tag{B.11}$$

Here the expression looks more complicated than necessary because the derivatives are taken with respect to λ.

Higher-order correlators are of increasingly cumbersome expression, but in principle they are easily computed using the general calculus outlined in the main text.

B.1. Conformal maps

A further simplification of the formulae arises in case the functions $\mathbf{Y}$ and $\mathbf{X}$ above describe the Riemann uniformization and its Schwartz reflected of a simply-connected domain $\mathcal{D}$ in the $\mathbf{X}$ plane. We recall that all our formulae can be easily adapted to the description of simply and multiply connected domains (the number of connected components being the genus of the curve) by taking the curve Σ_g as an M-curve in the sense of Harnack [20]: namely, a curve with an anti-holomorphic involution $\varphi : \Sigma_g \to \Sigma_g$ having $g+1$ contours of fixed points and such that

$$\mathbf{X}(\zeta) = \overline{\mathbf{Y}(\varphi(\zeta))}. \tag{B.12}$$

In genus zero and with the normalization used in the previous paragraph for the uniformizing coordinate, the anti-holomorphic involution would be $\lambda \to \frac{1}{\bar\lambda}$. The two functions $\mathbf{Y}$ and $\mathbf{X}$ then satisfy

$$\mathbf{X}(\lambda) = \overline{\mathbf{Y}\left(\frac{1}{\bar\lambda}\right)}. \tag{B.13}$$

Since $\mathbf{X}(\lambda)$ is now the uniformizing map of a simply-connected domain $\mathcal{D}$, it follows from the general properties of such maps that $\mathbf{X}$ maps biholomorphically the outer region $\mathbb{C}\setminus\mathcal{D}$ to the outside of the unit disk in the λ-plane. This means that the zeros of $\mathrm{d}\mathbf{X}$ all lie inside the unit disk and hence the zeros of $\mathrm{d}\mathbf{Y}$ (which is the Schwartz function of the domain) all lie outside.

The free energy of the two-matrix model under this reduction $v_K = \overline{u}_K$, reduces to the tau function of Jordan curves studied in [24, 25, 28, 31–33] as explained in [1, 2]

$$\mathcal{F} = \frac{1}{4\pi^2}\int_{\mathcal{D}}\int_{\mathcal{D}} \mathrm{d}^2\mathbf{X}\,\mathrm{d}^2\widetilde{\mathbf{X}}\ln\left|\frac{1}{\mathbf{X}}-\frac{1}{\widetilde{\mathbf{X}}}\right|. \tag{B.14}$$

[9] Note that the set of points $\{\lambda_{j,X}\}$ is a subset of the $\{x_k\}$ and similarly for the $\{\lambda_{j,Y}\}$ and $\{y_k\}$.

The coordinates u_K are then identified with the so-called *exterior harmonic moments* of the region

$$t = \frac{1}{2\mathrm{i}\pi}\int_{\mathcal{D}} \mathrm{d}\mathbf{X}\wedge \mathrm{d}\overline{\mathbf{X}} = \bar{t} \tag{B.15}$$

$$u_k = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}\setminus\mathcal{D}} \mathbf{X}^{-K}\,\mathrm{d}\mathbf{X}\wedge \mathrm{d}\overline{\mathbf{X}}, \tag{B.16}$$

and can be transformed in contour integrals along the boundary of the region $\mathcal{D}$ using Green's theorem (for $K \leqslant 2$ the *definition* of the exterior harmonic moments requires actually a regularization which is equivalent to replacing the surface integral by its corresponding boundary integral).

The free energy is in this case a real analytic function of the harmonic moments $\mathcal{F} = (u_K, \overline{u_K}, t)$ and the previous formulae for the fourth derivatives[10] can be translated into contour integral-formulae which in turn could be written in terms of Green's function of the Laplacian for the given region. It is also clear that effective formulae can be obtained for the multiply connected domains which correspond to higher genus M-curves considered in this context.

Appendix C. An extended Whitham moduli space

The moduli space considered in this paper could be easily extended in the spirit of [26] by considering instead of functions $\mathbf{X}$, $\mathbf{Y}$ some normalized second-kind differentials $\mathrm{d}\mathbf{X}$, $\mathrm{d}\mathbf{Y}$: this generalization has probably no relevance in the context of matrix models, nevertheless we sketch the main extra features. The practical difference is that now we may still think of *multivalued* functions $\mathbf{X}$, $\mathbf{Y}$ with the properties

$$\mathbf{X}(\zeta + b_j) = \mathbf{X}(\zeta) + A_j \tag{C.1}$$

$$\mathbf{Y}(\zeta + b_j) = \mathbf{Y}(\zeta) + B_j, \tag{C.2}$$

whereas the functions have no multivaluedness along the a-cycles. The rest of the description of the moduli space is exactly as in section 2. Note that this moduli space is 'larger' than the moduli space of [26] because we are also considering the position of some zeros of our primary differentials.

After dissection of the surface Σ_g along the chosen cycles $\{a_j, b_j\}_{j=1,\ldots,g}$ and along the fixed contours between the non-hard-edge poles, we obtain a simply-connected domain over which we will consider the functions $\mathbf{X} = \int \mathrm{d}\mathbf{X}, \mathbf{Y} = \int \mathrm{d}\mathbf{Y}$. In this domain the same asymptotics as in (2.3) are valid (where the 'potentials' are discontinuous across the cuts along which we have dissected the surface). The free energy (we should probably call it rather the 'tau' function) would be defined by the same formulae (3.1) except for the fact that the ϵ_j-derivatives should be replaced by the formulae below and we should consider the derivatives with respect to the extra moduli A_j, B_j

$$\mathbb{A}_j := \partial_{A_j}\mathcal{F} = \frac{1}{2\mathrm{i}\pi}\left(\oint_{a_j} \mathbf{Y}\mathbf{X}\,\mathrm{d}\mathbf{Y} - \frac{1}{2}\epsilon_j B_j\right) \tag{C.3}$$

$$\mathbb{B}_j := \partial_{B_j}\mathcal{F} = \frac{1}{2\mathrm{i}\pi}\left(\oint_{a_j} \mathbf{Y}\mathbf{X}\,\mathrm{d}\mathbf{X} + \frac{1}{2}\epsilon_j A_j\right) \tag{C.4}$$

[10] The third derivatives were computed in [33].

$$\Gamma_j := \partial_{\epsilon_j}\mathcal{F} = \frac{1}{2\mathrm{i}\pi}\left(\frac{1}{2}A_jB_j - B_j\mathbf{X}(\zeta) + \int_\zeta^{\zeta+b_j}\mathbf{Y}\,\mathrm{d}\mathbf{X}\right)$$
$$= -\frac{1}{2\mathrm{i}\pi}\left(\frac{1}{2}A_jB_j - A_j\mathbf{Y}(\zeta) + \int_\zeta^{\zeta+b_j}\mathbf{X}\,\mathrm{d}\mathbf{Y}\right). \qquad \text{(C.5)}$$

The equivalence of the two last lines is given by integration by parts. Also, the last integrals may seem to depend on the base point of integration: in fact they do not as one may check by computing the differential at ζ.

Besides the differentials considered in (3.13, 3.14 and 3.15), one also has

$$\left(\partial_{A_j}\mathbf{X}\right)_{\mathbf{Y}}\mathrm{d}\mathbf{Y} = \frac{1}{2\mathrm{i}\pi}\oint_{a_j}\mathbf{Y}\Omega =: -\mathcal{A}_j(\Omega) \qquad \left(\partial_{B_j}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X} = \frac{1}{2\mathrm{i}\pi}\oint_{a_j}\mathbf{X}\Omega =: \mathcal{B}_j(\Omega). \qquad \text{(C.6)}$$

These formulae are obtained by noticing that $\left(\partial_{B_j}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X}$ is a holomorphic *multivalued* differential with monodromy only around the corresponding b-cycle

$$\left(\partial_{B_j}\mathbf{Y}\right)_{\mathbf{X}}\mathrm{d}\mathbf{X}\Big|_\zeta^{\zeta+b_k} = -\delta_{jk}\,\mathrm{d}\mathbf{X}(\zeta). \qquad \text{(C.7)}$$

The integral formula has the same properties and hence we have the equality. The reasoning for $\left(\partial_{A_j}\mathbf{X}\right)_{\mathbf{Y}}\mathrm{d}\mathbf{Y}$ is symmetric. Note that—using the thermodynamic identity—we have

$$\left(\partial_{B_j}\mathbf{X}\right)_{\mathbf{Y}}\mathrm{d}\mathbf{Y} = -\frac{1}{2\mathrm{i}\pi}\oint_{a_j}\mathbf{X}\Omega. \qquad \text{(C.8)}$$

The considerations to prove the compatibility of the above equations are similar to the previous case with one notable exception we want to bring to the attention of the reader; in the computations of the second derivatives one is lead to considering integrals of the form

$$\oint_{a_j}\mathbf{X}\oint_{a_k}\mathbf{Y}\Omega, \qquad \oint_{a_j}\mathbf{X}\oint_{a_k}\mathbf{X}\Omega, \qquad \oint_{a_j}\mathbf{Y}\oint_{a_k}\mathbf{Y}\Omega. \qquad \text{(C.9)}$$

These integrals do not depend on the order only if $j \neq k$: in fact we have

$$\oint_{a_j}\mathbf{X}\oint_{a_j}\mathbf{Y}\Omega = \oint_{a_j}\mathbf{Y}\oint_{a_j}\mathbf{X}\Omega + 2\mathrm{i}\pi\oint_{a_j}\mathbf{X}\,\mathrm{d}\mathbf{Y}$$
$$\oint_{a_j}\mathbf{X}\oint_{a_k}\mathbf{X}\Omega = \oint_{a_j}\mathbf{X}\oint_{a_j}\mathbf{X}\Omega + 2\mathrm{i}\pi\oint_{a_j}\mathbf{X}\,\mathrm{d}\mathbf{X} = \oint_{a_j}\mathbf{X}\oint_{a_j}\mathbf{X}\Omega \qquad \text{(C.10)}$$
$$\oint_{a_j}\mathbf{Y}\oint_{a_j}\mathbf{Y}\Omega = \oint_{a_j}\mathbf{Y}\oint_{a_j}\mathbf{Y}\Omega + 2\mathrm{i}\pi\oint_{a_j}\mathbf{Y}\,\mathrm{d}\mathbf{Y} = \oint_{a_j}\mathbf{Y}\oint_{a_j}\mathbf{Y}\Omega.$$

Another kind of integrals that one encounters are of the type

$$\oint_{a_j}\mathbf{Y}\oint_{b_k}\Omega = 2\mathrm{i}\pi\oint_{a_j}\mathbf{Y}\omega_k. \qquad \text{(C.11)}$$

Here one has to use the following rule for exchanging the order of the integrals: suppose that a specific choice of the homology representatives of a_j and b_j intersect at the point ζ_0, then

$$\oint_{\zeta\in a_j}F(\zeta)\oint_{\xi\in b_k}\Omega(\zeta,\xi) = \oint_{\zeta\in a_j}(F(\zeta)-F(\zeta_0))\oint_{\xi\in b_k}\Omega(\zeta,\xi) + F(\zeta_0)\oint_{\zeta\in a_j}\oint_{\xi\in b_k}\Omega(\zeta,\xi) \qquad \text{(C.12)}$$
$$= \oint_{\xi\in b_k}\oint_{\zeta\in a_j}(F(\zeta)-F(\zeta_0))\Omega(\zeta,\xi) - 2\mathrm{i}\pi\delta_{jk}F(\zeta_0)$$
$$= 2\mathrm{i}\pi\delta_{jk}F(\zeta_0) + \oint_{\xi\in b_k}\oint_{\zeta\in a_j}F(\zeta)\Omega(\zeta,\xi). \qquad \text{(C.13)}$$

Following similar arguments used in section 3, one can prove that

$$2\mathcal{F} = 2\mathcal{F}_0 + \sum_{j=1}^{g} (A_j \mathbb{A}_j + B_j \mathbb{B}_j), \tag{C.14}$$

where $\mathcal{F}_0$ is given by the same formula (3.25) (with the new meaning of Γ_j, though). The proof rests on the identity

$$\mathbf{Y}\,\mathrm{d}\mathbf{X} = \sum_{\alpha=0} \sum_{K=0}^{d_{1,\alpha}} u_{K,\alpha} \mathcal{U}_{K,\alpha}(\Omega) + \sum_{\alpha=0} \sum_{J=0}^{d_{2,\alpha}} v_{J,\alpha} \mathcal{V}_{J,\alpha}(\Omega) \tag{C.15}$$

$$+ t\mathcal{T}(\Omega) + \sum_{j=1}^{g} \left(\epsilon_j \mathcal{E}_j(\Omega) + \frac{1}{2\mathrm{i}\pi} \oint_{a_j} \mathbf{X}\Omega + \frac{1}{2\mathrm{i}\pi} \oint_{a_j} \mathbf{Y}\Omega \right) + \sum_{\zeta \in \mathcal{D}_{\mathbf{Y}}} \operatorname*{res}_{\zeta} \mathbf{X}\mathbf{Y}\Omega, \tag{C.16}$$

which is proved as before by matching the singular behaviours of both sides at all possible singularities and by checking that both sides have the same multivaluedness around the a- and b-cycles and the same periods.

C.1. Higher-order derivatives

In order to write compactly the second derivatives, let us denote by ∂ any derivative with respect to one of the parameters $u_{K,\alpha}, Q_\alpha, X_j, v_{J,\alpha}, P_\alpha, Y_j$. Beside the second derivatives already computed, the new ones are given by the formulae

$$\begin{aligned}
&\partial_{A_j}\partial_{A_k}\mathcal{F} = \mathcal{A}_j\mathcal{A}_k\Omega, \qquad \partial_{B_j}\partial_{B_k}\mathcal{F} = \mathcal{B}_j\mathcal{B}_k\Omega \\
&\partial_{A_j}\partial_{B_k}\mathcal{F} = \mathcal{A}_j\mathcal{B}_k\Omega + \frac{\delta_{jk}}{4\mathrm{i}\pi}\epsilon_k \qquad \partial_{A_j}\partial_{\epsilon_k}\mathcal{F} = \mathcal{A}_j\mathcal{E}_k\Omega + \frac{\delta_{jk}}{4\mathrm{i}\pi}B_j \\
&\partial_{B_j}\partial_{\epsilon_k}\mathcal{F} = \mathcal{B}_j\mathcal{E}_k\Omega - \frac{\delta_{jk}}{4\mathrm{i}\pi}A_k \qquad \partial_{B_j}\partial\mathcal{F} = \mathcal{B}_j\int_\partial \Omega, \qquad \partial_{A_j}\partial\mathcal{F} = \mathcal{A}_j\int_\partial\Omega.
\end{aligned} \tag{C.17}$$

We remark that the order of the integral operators acting on Ω is relevant because Ω is singular on the diagonal: for instance,

$$\mathcal{A}_j\mathcal{B}_k\Omega = \mathcal{B}_k\mathcal{A}_j\Omega - \frac{\delta_{jk}}{2\mathrm{i}\pi}\epsilon_k. \tag{C.18}$$

In order to compute all higher derivatives and loop correlators, we need to specify the relevant additional Rauch variational formulae: besides those considered in (3.53), we need those related to the extra moduli

$$\begin{aligned}
\left(\partial_{A_j}\Omega\right)_{\mathbf{X}}(1,2) &= -\frac{1}{2\mathrm{i}\pi}\oint_{\xi\in a_j}\left(\mathbf{Y}(\xi)\Omega^{(3)}_{\mathbf{X}}(1,2,\xi) - \frac{\Omega(1,\xi)\Omega(2,\xi)}{\mathrm{d}\mathbf{X}(\xi)}\right) \\
\left(\partial_{B_j}\Omega\right)_{\mathbf{X}}(1,2) &= \frac{1}{2\mathrm{i}\pi}\oint_{\xi\in a_j}\mathbf{X}(\xi)\Omega^{(3)}_{\mathbf{X}}(1,2,\xi) \\
\left(\partial_{B_j}\Omega\right)_{\mathbf{Y}}(1,2) &= \frac{1}{2\mathrm{i}\pi}\oint_{\xi\in a_j}\left(\mathbf{X}(\xi)\Omega^{(3)}_{\mathbf{Y}}(1,2,\xi) + \frac{\Omega(1,\xi)\Omega(2,\xi)}{\mathrm{d}\mathbf{Y}(\xi)}\right) \\
\left(\partial_{A_j}\Omega\right)_{\mathbf{Y}}(1,2) &= -\frac{1}{2\mathrm{i}\pi}\oint_{\xi\in a_j}\mathbf{Y}(\xi)\Omega^{(3)}_{\mathbf{Y}}(1,2,\xi).
\end{aligned} \tag{C.19}$$

We briefly justify these formulae. Suppose ω is any of our differentials and consider the function $\omega/\mathrm{d}\mathbf{X}$ (or symmetric argument for $\mathbf{Y}$). This function has poles at the zeros of $\mathrm{d}\mathbf{X}$ and possibly constant monodromy around a b-cycle. Thinking of it as a function of $\mathbf{X}$, the monodromy condition reads (c is 0 or 1 depending on the case chosen, but the argument is unaffected)

$$\frac{\omega}{\mathrm{d}\mathbf{X}}(\mathbf{X}+A_j)-\frac{\omega}{\mathrm{d}\mathbf{X}}(\mathbf{X})=c. \tag{C.20}$$

Taking the derivative with respect to A_j at $\mathbf{X}$-fixed, we have

$$\left(\partial_{A_j}\omega\right)_{\mathbf{X}}\Big|_{\zeta}^{\zeta+b_j}=\mathrm{d}\left(\frac{\omega}{\mathrm{d}\mathbf{X}}\right). \tag{C.21}$$

Considering with some care, the singularities at the zeros of $\mathrm{d}\mathbf{X}$ and this multivaluedness, one gets

$$\left(\partial_{A_j}\omega\right)(\zeta)=-\sum\operatorname*{res}_{\xi=x_k}\frac{\omega(\xi)\Omega(\xi,\zeta)\mathcal{A}_j(\Omega)(\xi)}{\mathrm{d}\mathbf{X}(\xi)\,\mathrm{d}\mathbf{Y}(\xi)}+\frac{1}{2\mathrm{i}\pi}\oint_{a_j}\frac{\Omega(\zeta,\xi)\omega(\xi)}{\mathrm{d}\mathbf{X}(\xi)}. \tag{C.22}$$

This gives the previous extended Rauch formulae.

Using these expressions for the variation of the Bergman kernel, one can obtain all third derivatives. Besides those already considered in (3.56, 3.57 and 3.58), we also find

$$\begin{aligned}
&\partial_{A_j}\partial_{A_k}\partial_{A_\ell}\mathcal{F}=\mathcal{A}_j\mathcal{A}_k\mathcal{A}_\ell\Omega^{(3)}_{\mathbf{Y}}; \qquad \partial_{B_j}\partial_{B_k}\partial_{B_\ell}\mathcal{F}=\mathcal{B}_j\mathcal{B}_k\mathcal{B}_\ell\Omega^{(3)}_{\mathbf{X}}\\
&\partial_{A_j}\partial_{A_k}\partial\mathcal{F}=\int_\partial\mathcal{A}_j\mathcal{A}_k\Omega^{(3)}_{\mathbf{Y}}; \qquad \partial_{B_j}\partial_{B_k}\partial\mathcal{F}=\int_\partial\mathcal{B}_j\mathcal{B}_k\Omega^{(3)}_{\mathbf{X}}\\
&\partial_{A_j}\partial_{B_k}\partial_{\epsilon_\ell}\mathcal{F}=\mathcal{A}_j\mathcal{B}_k\mathcal{E}_\ell\Omega^{(3)}_{\mathbf{X}}+\frac{1}{2\mathrm{i}\pi}\oint_{a_j}\frac{\mathcal{B}_k(\Omega)\mathcal{E}_j(\Omega)}{\mathrm{d}\mathbf{X}}-\frac{\delta_{jk}\delta_{kl}}{4\mathrm{i}\pi}\\
&\partial_{A_j}\partial_{B_k}\partial_{v_{J,\alpha}}\mathcal{F}=\mathcal{A}_j\mathcal{B}_k\mathcal{V}_{J,\alpha}\Omega^{(3)}_{\mathbf{Y}}+\frac{1}{2\mathrm{i}\pi}\oint_{a_k}\frac{\mathcal{A}_j(\Omega)\mathcal{V}_{J,\alpha}(\Omega)}{\mathrm{d}\mathbf{Y}}\\
&\partial_{A_j}\partial_{B_k}\partial_{u_{K,\alpha}}\mathcal{F}=\mathcal{A}_j\mathcal{B}_k\mathcal{U}_{K,\alpha}\Omega^{(3)}_{\mathbf{X}}+\frac{1}{2\mathrm{i}\pi}\oint_{a_j}\frac{\mathcal{B}_k(\Omega)\mathcal{U}_{K,\alpha}(\Omega)}{\mathrm{d}\mathbf{X}}\\
&\partial_{A_j}\partial_{u_{K,\alpha}}\partial_{v_{J,\beta}}\mathcal{F}=\mathcal{A}_j\mathcal{U}_{K,\alpha}\mathcal{V}_{J,\beta}\Omega^{(3)}_{\mathbf{Y}}; \qquad \partial_{B_j}\partial_{u_{K,\alpha}}\partial_{v_{J,\beta}}\mathcal{F}=\mathcal{B}_j\mathcal{U}_{K,\alpha}\mathcal{V}_{J,\beta}\Omega^{(3)}_{\mathbf{X}}\\
&\partial_{A_j}\partial_{u_{K,\alpha}}\partial_{u_{J,\beta}}\mathcal{F}=\mathcal{A}_j\mathcal{U}_{K,\alpha}\mathcal{U}_{J,\beta}\Omega^{(3)}_{\mathbf{X}}+\frac{1}{2\mathrm{i}\pi}\oint_{a_j}\frac{\mathcal{U}_{K,\alpha}(\Omega)\mathcal{U}_{J,\beta}(\Omega)}{\mathrm{d}\mathbf{X}}\\
&\partial_{B_j}\partial_{v_{K,\alpha}}\partial_{v_{J,\beta}}\mathcal{F}=\mathcal{B}_j\mathcal{V}_{K,\alpha}\mathcal{V}_{J,\beta}\Omega^{(3)}_{\mathbf{Y}}+\frac{1}{2\mathrm{i}\pi}\oint_{a_j}\frac{\mathcal{V}_{K,\alpha}(\Omega)\mathcal{V}_{J,\beta}(\Omega)}{\mathrm{d}\mathbf{Y}}\\
&\partial_{A_j}\partial_{v_{K,\alpha}}\partial_{v_{J,\beta}}\mathcal{F}=\mathcal{A}_j\mathcal{V}_{K,\alpha}\mathcal{V}_{J,\beta}\Omega^{(3)}_{\mathbf{Y}}; \qquad \partial_{B_j}\partial_{u_{K,\alpha}}\partial_{u_{J,\beta}}\mathcal{F}=\mathcal{B}_j\mathcal{U}_{K,\alpha}\mathcal{U}_{J,\beta}\Omega^{(3)}_{\mathbf{X}}\\
&\partial_{A_j}\partial_t\partial\mathcal{F}=\int_\partial\mathcal{T}\mathcal{A}_j\Omega^{(3)}_{\mathbf{Y}}; \qquad \partial_{B_j}\partial_t\partial\mathcal{F}=\int_\partial\mathcal{T}\mathcal{B}_j\Omega^{(3)}_{\mathbf{X}}.
\end{aligned} \tag{C.23}$$

C.1.1. Order 4 and higher. It is clear that the formulae become rather long due to many-case distinctions. However, the reader should be able to compute any derivative of order 4 or higher by using the same rules of calculus outlined in the main text, with the additional Rauch formulae (C.19).

Appendix D. General definition of regularized integrals

Let ω be a meromorphic differential with poles at the points ζ_ρ, $\rho=0,\ldots$. Let z_ρ be chosen and fixed local parameters at ζ_ρ. Let ω_j be the Abelian differentials of the first kind normalized with respect to the a-cycles of a given choice of basis $\{a_j,b_j\}$ in the homology of the curve. Then we have

$$\omega=\sum_{\rho\geqslant 0}\sum_{K\geqslant 1}\frac{1}{K}\operatorname*{res}_{\zeta_\rho}(z_\rho)^K\operatorname*{res}_{\zeta_\rho}(z_\rho)^{-K}\Omega+\sum_{\rho\geqslant 1}\left(\operatorname*{res}_{\zeta_\rho}\omega\right)\int_{\zeta_0}^{\zeta_\rho}\Omega+\sum_{j=1}^{g}\left(\oint_{a_j}\omega\right)\omega_j. \tag{D.1}$$

The regularized integral from ξ to η is defined for a homology class of contours in the punctured surface: in general, one has to dissect the surface along the a, b-cycles and along a set of mutually non-intersecting segments joining the poles of ω in such a way as to have a simply-connected domain. Choosing an arbitrary path within this simply-connected region and joining the two chosen points we have (supposing that both ξ, η are poles of ω)

$$\begin{aligned}\fint_\xi^\eta \omega = \sum_{\rho\geqslant 0}\sum_{K\geqslant 1}\frac{1}{K}\operatorname*{res}_{\zeta_\rho}(z_\rho)^K \operatorname*{res}_{\zeta_\rho}(z_\rho)^{-K}\frac{\mathrm{d}\Lambda}{\Lambda} + \sum_{\substack{\rho\geqslant 0\\ \zeta_\rho\notin\{\xi,\eta\}}}\left(\operatorname*{res}_{\zeta_\rho}\omega\right)\ln\left(\frac{\Lambda(\zeta_\rho)}{\gamma_\xi}\right)\\ + \left(\operatorname*{res}_{\eta}\omega\right)\ln\left(\frac{\gamma_\eta}{\gamma_\xi}\right) + \sum_{j=1}^{g}\left(\oint_{a_j}\omega\right)\oint_{b_j}\frac{\mathrm{d}\Lambda}{\Lambda}\end{aligned} \tag{D.2}$$

$$\Lambda := \exp\left(\int\!\!\int_\xi^\eta \Omega\right) : \frac{\mathrm{d}\Lambda}{\Lambda} := \int_\xi^\eta \Omega \qquad \gamma_\xi := \lim_{\epsilon\to\xi}\ln\left(\frac{\Lambda(\epsilon)}{z_\xi(\epsilon)}\right) \tag{D.3}$$

$$\gamma_\eta := \lim_{\epsilon\to\eta}\ln(\Lambda(\epsilon)z_\eta(\epsilon)). \tag{D.4}$$

Some remarks are in order: the function $\ln(\Lambda)$ is defined as any antiderivative of the normalized third kind differential $\int_\xi^\eta \Omega$, which has residue -1 at ξ and residue $+1$ at η. Hence Λ has a simple zero at ξ and a simple pole at η (in the simply-connected domain). Also, Λ is defined up to a multiplicative constant depending on the base point of integration: the final formula for the regularized integral does not depend on this constant. Λ can be written explicitly in terms of a theta function and the b-periods of $\frac{\mathrm{d}\Lambda}{\Lambda}$ are the difference of the Abel map between ξ and η.

In the more general situation of the extended moduli space studied in appendix C, we had also some multivaluedness of the type

$$\omega(\zeta + b_j) - \omega(\zeta) = \mathrm{d}H_j(\zeta), \tag{D.5}$$

where $\mathrm{d}H_j(\zeta)$, $j = 1, \ldots, g$, are meromorphic differential of the second kind with vanishing a-cycles. The formula for a regularized integral is easily adapted: the main observation is that (D.1) now needs on the rhs, the following extra term:

$$\omega = (\mathrm{D}.1) + \frac{1}{2\mathrm{i}\pi}\sum_{j=1}^{g}\oint_{a_j} H_j\Omega \tag{D.6}$$

and consequently the formula for the regularized integral is

$$\fint_\xi^\eta \omega = (\mathrm{D}.2) + \frac{1}{2\mathrm{i}\pi}\sum_{j=1}^{g}\oint_{a_j} H_j\frac{\mathrm{d}\Lambda}{\Lambda}. \tag{D.7}$$

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